A note on the degree condition of completely independent spanning trees

Hung-Yi Chang¹ Hung-Lung Wang¹ Jinn-Shyong Yang² Jou-Ming Chang¹†

¹ Institute of Information and Decision Sciences, National Taipei University of Business, Taipei, Taiwan, ROC
² Department of Information Management, National Taipei University of Business, Taipei, Taiwan, ROC

Abstract

Given a graph $G$, a set of spanning trees of $G$ are completely independent if for any vertices $x$ and $y$, the paths connecting them on these trees have neither vertex nor edge in common, except $x$ and $y$. In this paper, we prove that for graphs of order $n$, with $n \geq 6$, if the minimum degree is at least $n-2$, then there are at least $\lceil n/3 \rceil$ completely independent spanning trees.

Keyword: Completely independent spanning trees; Dirac's condition; Ore's condition;

1 Introduction

In a graph $G$, a set of spanning trees are said to be completely independent if for any vertices $x$ and $y$, the paths connecting them on the spanning trees have neither vertex nor edge in common, except $x$ and $y$. Graphs discussed in this paper are assumed to be connected and simple, where a graph is simple if it contains neither parallel edge nor loop. The concept of completely independent spanning trees was introduced by Hasunuma [3] in 2001. He showed that determining whether a graph admits $k$ completely independent spanning trees is NP-complete, even for $k = 2$ [4]. The construction of $k$ completely independent spanning trees was then investigated on some graph classes, with connectivity at least $2k$ [3–5]. In addition, Hasunuma [4] gave a conjecture, which states that being $2k$-connected is sufficient for having $k$ completely independent spanning trees, but that was disproved by Péterfalve [7]. Counterexamples were also given by Pai et al. [6], via a necessary condition to a $k$-connected and $k$-regular graph with $\lceil k/2 \rceil$ completely independent spanning trees.

In 2013, Araki [1] gave a new characterization, which is summarized later in Section 2, for graphs admitting $k$ completely independent spanning trees. Based on this, he proved that the following condition is sufficient for a graph $G$, with enough vertices, admitting two completely independent spanning trees.

\[ \forall x \in V \ deg(x) \geq n/2, \] (1)

where $V$ is the vertex set of $G$, $n = |V|$, and $\deg(x)$ is the degree of $x$. Usually, (1) is called the Dirac’s condition, due to the well-known sufficient condition for a graph being Hamiltonian. Later on, Fan et al. [2] generalized the result by Ore’s condition, which is

\[ \forall x, y \in V \ deg(x) + deg(y) \geq n/2. \] (2)

Both (1) and (2) deal with the existence of two completely independent spanning trees via the constraint on vertex degree. This motivates us to relate the number of completely independent spanning trees and some possible constraints on vertex degree.

The main result of this paper is the following theorem.

Theorem 1. Let $G$ be a graph of order $n$ with minimum degree at least $n-2$. If $n \geq 6$, there are $\lceil n/3 \rceil$ completely independent spanning trees.

The proof of Theorem 1 is given in Section 3, and some preliminaries are given in Section 2. In the discussion below, we assume that $V$ is the vertex set of the graph being considered and $|V| = n \geq 6$. 

*The authors were supported in part by MOST grants 103-2221-E-141-001 (Hung-Yi Chang, Jou-Ming Chang), 103-2221-E-141-004 (Hung-Lung Wang), and 103-2221-E-141-003 (Jinn-Shyong Yang), from the Ministry of Science and Technology, Taiwan.

†Corresponding author. Email: spade@ntub.edu.tw
2 Preliminaries

In a graph $G$, the minimum degree $\delta(G)$ of $G$ is defined as $\min\{\deg(x) \mid x \in V\}$. A vertex $x$ is said to be universal if $\deg(x) = n - 1$. The open neighborhood of a vertex $x$ consists of all the vertices adjacent to $x$, denoted by $N(x)$. We use $P_i$ and $C_i$ to denote a path and a cycle with $i$ vertices, respectively. For $U \subseteq V$, the subgraph induced by $U$ is denoted by $G[U]$. For $x \in V$, we denote the subgraph induced by $V \setminus \{x\}$ by $G - x$, and analogously, we use $G - U$ for $G[V \setminus U]$. For an edge with vertices $x$ and $y$, we denote it by $xy$, and the graph obtained by removing $xy$ is denoted by $G - xy$. Let $E$ be the edge set of $G$. For disjoint subsets $U_1, U_2 \subseteq V$, the graph $B(U_1, U_2)$ is defined to be a bipartite graph with partite sets $U_1$ and $U_2$, and edge set $\{xy \in E \mid x \in U_1, y \in U_2\}$. Araki [1] gave the following characterization for a graph admitting $k$ completely independent spanning trees.

**Theorem 2** (See [1]). A graph $G$ has $k$ completely independent spanning trees if and only if there is a partition of $V$ into $V_1, V_2, \ldots, V_k$ such that

- for $i \in \{1, 2, \ldots, k\}$, $G[V_i]$ is connected.
- for distinct $i, j \in \{1, 2, \ldots, k\}$, $B(V_i, V_j)$ has no tree component.

The partition is called a CIST-partition of $G$.

3 Main results

To simply the presentation, we elaborate the results for even $n$ and odd $n$, separately, in Sections 3.1 and 3.2. The proofs for both cases are very similar.

3.1 For even $n$

The idea for proving Theorem 1 for a graph $G$ of even order is to show the following.

- There is an $(n - 2)$-regular spanning subgraph of $G$.
- All $(n - 2)$-regular graphs are isomorphic.
- There are $\lfloor n/3 \rfloor$ completely independent spanning trees in an $(n - 2)$-regular graph.

Details are given in Lemmas 1, 2, and 3.

**Lemma 1.** Let $G_1$ and $G_2$ be simple graphs of order $n$, where $n$ is even. If $G_1$ and $G_2$ are $(n - 2)$-regular, then $G_1$ and $G_2$ are isomorphic.

**Proof.** Let $n = 2k$, for $k \geq 1$. We prove this lemma by induction on $k$. For $k = 1$ the lemma holds trivially. Consider the case where $k > 1$. Since $G_1$ is $(2k - 2)$-regular, there are two vertices $x_1$ and $x_2$ which are not adjacent. It follows immediately that $N(x_1) = N(x_2)$, and $G_1[N(x_1)]$ is a $(2k - 4)$-regular graph. Similarly, for $G_2$, there are two non-adjacent vertices $y_1$ and $y_2$ such that $G_2[N(y_1)]$ is $(2k - 4)$-regular. By the induction hypothesis, $G_1[N(x_1)]$ and $G_2[N(y_1)]$ are isomorphic, and let $f: V(G_1[N(x_1)]) \rightarrow V(G_2[N(y_1)])$ be an isomorphism. It can be easily derived that $G_1$ and $G_2$ are isomorphic via the isomorphism $f': V(G_1) \rightarrow V(G_2)$, defined as

$$f'(v) = \begin{cases} y_1 & \text{if } v = x_1, \\ y_2 & \text{if } v = x_2, \\ f(v) & \text{otherwise.} \end{cases}$$

This proves the lemma. \qed

**Lemma 2.** Let $n$ be a positive even integer, and let $G$ be a simple graph with $\delta(G) \geq n - 2$. There exists an $(n - 2)$-regular spanning subgraph of $G$.

**Proof.** Since $\delta(G) \geq n - 2$, we may assume that there are $i$ vertices with degree $n - 1$ and $n - i$ vertices with degree $n - 2$, where $0 \leq i \leq n$. Considering the sum of vertex degrees of $G$, we have

$$\sum_{v \in V(G)} \deg(v) = i(n - 1) + (n - i)(n - 2) = n(n - 2) + i,$$

which leads to the fact that $i$ is even. For $i \neq 0$, there are at least two adjacent vertices which are of degree $n - 1$. By iteratively removing these edges, we have an $(n - 2)$-regular subgraph. \qed

**Remark 1.** If $n$ is even, the $(n - 2)$-regular graph can be obtained by removing a perfect matching from $K_n$.

**Remark 2.** Graphs with minimum degree $k$ do not always contain a $k$-regular subgraph, even for $k = n - 3$.

**Lemma 3.** Let $G$ be an $(n - 2)$-regular graph. If $n \geq 6$, then there are $\lfloor n/3 \rfloor$ completely independent spanning trees.

**Proof.** By Theorem 2, it suffices to show that there is a size $\lfloor n/3 \rfloor$ CIST-partition. Let $k = \lfloor n/3 \rfloor$, and let $V$ be partitioned into $V_1, V_2, \ldots, V_k$ as even as possible, i.e.,

$$\forall i \in \{1, \ldots, k\} \quad 3 \leq |V_i| \leq 4.$$ 

Without loss of generality, let $j$ be the index such that $|V_i| = 3$ for $1 \leq i \leq j$, and $|V_i| = 4$ for $i > j$.  

48
An illustration is given in Figure 1. By definition, one vertex

detail as follows. A straightforward manner. We elaborate in more

lemmas in Section 3.1 hold. Therefore, the “odd

graph

$G$

$j$

The CIST-partition of

Figure 1: The CIST-partition of $G$ defined in Lemma 3. The vertices are pairwise adjacent, except those connected by dotted lines. In other words, dotted lines denoted the perfect matching removed from $K_n$.

Notice that when $j = 0$, there is no $V_i$ of size 3. Clearly, $j$ is even, and we can further define the partition so that

- $G[V_i]$ is $P_3$, for $1 \leq i \leq j$.
- $G[V_{2i-1} \cup V_{2i}]$ is a 4-regular graph, for $1 \leq i \leq j/2$.
- $G[V_i]$ is $C_4$, for $i > j$.

An illustration is given in Figure 1. By definition, $V_i$ induces either $P_3$ or $C_4$ on $G$. Moreover, for any two subsets $V_i$ and $V_j$, $B(V_i, V_j)$ is either $K_{3,3} - e$ or a complete bipartite graph other than a star. It follows that $V_1, V_2, \ldots, V_k$ is a CIST-partition of $G$. This proves the lemma.

3.2 For odd $n$

For odd $n$, any graph $G$ with minimum degree at least $n - 2$ cannot be $(n-2)$-regular since otherwise $\sum_v \deg(v)$, which is $n(n-2)$, is odd. Thus, at least one vertex $x$ of $G$ has degree $n - 1$. Consider the graph $G - x$. Obviously, $G - x$ is a graph on which lemmas in Section 3.1 hold. Therefore, the “odd $n$ version” of Lemmas 1 and 2 can be derived in a straightforward manner. We elaborate in more detail as follows.

For convenience, we define a graph $G$ to be $k^+$-regular if

- there is exactly one vertex $x$ with $\deg(x) = k + 1$, and
- for each vertex $y$ other than $x$, $\deg(y) = k$.

Lemma 4. Let $G_1$ and $G_2$ be simple graphs of order $n$, where $n$ is odd. If $G_1$ and $G_2$ are $(n-2)^+$-regular, then $G_1$ and $G_2$ are isomorphic.

Lemma 5. Let $n$ be a positive odd integer, and let $G$ be a simple graph with $\delta(G) \geq n-2$. There exists an $(n-2)^+$-regular spanning subgraph of $G$.

Similar to Lemma 3, we can show the existence of $\lceil n/3 \rceil$ completely independent spanning trees of $G$ by defining a CIST-partition of size $\lceil n/3 \rceil$. The partition is similar to that defined in Lemma 3. The only modification we need is to deal with the odd number of subsets of size three. By letting the universal vertex be in one of these subsets, we get the requested CIST-partition. More precisely, let $x$ be the universal vertex, and let $yz$ be a non-edge. The graph $G - \{x, y, z\}$ is of order $n - 3$ and is $(n - 5)$-regular. We can partition the vertices of $G - \{x, y, z\}$ as we do in Lemma 3. Together with the subset $\{x, y, z\}$, we have the CIST-partition of $G$. This yields the following theorem.

Lemma 6. Let $G$ be an $(n-2)^+$-regular graph with order $n$. If $n \geq 6$, then there are $\lceil n/3 \rceil$ completely independent spanning trees in $G$.

3.3 Proof of Theorem 1

Proof. Consider the case where $n$ is even. By Lemmas 1 and 2, it suffices to show that “the” $(n-2)$-regular graph contains $\lceil n/3 \rceil$ CISTs, and the result is given by Lemma 3. Similarly, for $n$ being odd, the result can be derived from Lemmas 4, 5 and 6. Thus, Theorem 1 is proved.

4 Concluding remarks

In this paper, we show that for a graph $G$ of order $n \geq 6$ with $\delta(G) \geq n - 2$, there are $\lfloor n/3 \rceil$ completely independent spanning trees. As mentioned in Section 1, the following results are known:

- If $\delta(G) \geq n-1$, there are $\lfloor (n-1)/2 \rceil$ completely independent spanning trees [6].
- If $\delta(G) \geq n/2$, there are 2 completely independent spanning trees [1].

A straightforward generalization of the above two statements is “If $\delta(G) \geq a$, there are $b$ completely independent spanning trees.” How $a$ and $b$ are related is what we are interested in, and this problem will be conducted as future work.
References