Approximation algorithms for single allocation k-hub center problem*

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Abstract

Given a metric graph $G = (V, E, w)$ and a positive integer k, the Single Allocation k-Hub Center *problem is to find a spanning subgraph H^* of G such that (i) $C^* \subset V$ is a clique of size k in H^*; (ii) $V \setminus C^*$ forms an independent set in H^*; (iii) each $v \in V \setminus C^*$ is adjacent to exactly one vertex in C^*; and (iv) the diameter $D(H^*)$ is minimized. The vertices selected in C^* are called* hubs *and the rest of vertices are called* non-hubs. *The* Single Allocation k-Hub Center *problem is NP-hard in metric graphs. In this paper, we show that for any $\epsilon > 0$, it is NP-hard to approximate the* Single Allocation k-Hub Center *problem to a ratio $\frac{4}{3} - \epsilon$. Moreover, we give two approximation algorithms for solving this problem. One is a 2-approximation algorithm running in time $O(n)$ and the other is a $\frac{5}{3}$-approximation algorithm running in time $O(kn^3)$.*

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1 Introduction

The Single Allocation k-Hub Center problem is to choose a fixed number k of vertices as hubs and to assign each non-hub vertex to exactly one of the chosen hubs in such a way that the maximum distance/cost between origin-destination pairs is minimized. It has applications in airline [3] and cargo delivery systems [7]. The Single Allocation k-Hub Center problem is introduced in [7, 2]. Unlike the goal of classical hub location problems is to minimize the total cost of all origin-destination pairs (see *e.g.,* [10]), the Single Allocation k-Hub Center problem is to minimize the poorest service quality. The minmax criterion of the Single Allocation k-Hub Center is able to avoid the drawback that sometimes minimizing the total cost would lead to the result that the poorest service quality is extremely bad.

We consider a graph $G = (V, E, w)$ with a distance function $w(\cdot, \cdot)$ being a metric on V such that $w(v, v) = 0$, $w(u, v) = w(v, u)$, and $w(u, v) + w(v, r) \geq w(u, r)$ for all $u, v, r \in V$.

Single Allocation k-Hub Center (SAkHC)
Input: A metric graph $G = (V, E, w)$ and a positive integer k.
Output: A spanning subgraph H^* of G such that (i) vertices (hubs) in $C^* \subset V$ form a clique of size k in H^*; (ii) vertices (non-hubs) in $V \setminus C^*$ form an independent set in H^*; (iii) each non-hub $v \in V \setminus C^*$ is adjacent to exactly one hub in C^*; and (iv) the diameter $D(H^*)$ is minimized.

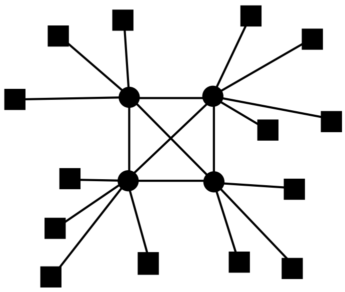

Figure 1: An example of single allocation k-hub center networks with $k = 4$ where the circle nodes and the square nodes denote hubs and non-hubs, respectively.

The SINGLE ALLOCATION k-HUB CENTER problem is NP-hard in metric graphs [4]. Several linearizations of the integer program and quadratic program were proposed in the literature [2, 4, 3, 6]. Many research efforts for solving the SINGLE ALLOCATION k-HUB CENTER problem are focused on the development of heuristic algorithms, *e.g.,* [8, 11, 12, 13, 9, 1].

In this paper, we investigate the approximability of the SINGLE ALLOCATION k-HUB CENTER problem. The paper is organized as follows: In Section 2, we prove that for any $\epsilon > 0$, it is NP-hard to approximate the SINGLE ALLOCATION k-HUB CENTER problem to a ratio $\frac{4}{3} - \epsilon$. In Section 3, we give a 2-approximation algorithm running in time $O(n)$ for the SINGLE ALLOCATION k-HUB CENTER problem where n is the number of vertices in the input graph. In Section 4, we give a 5/3-approximation algorithm for the same problem running in time $O(kn^3)$.

We close this section with some notation definitions. For a vertex v in a graph H, we use $N_H(v)$ to denote the set of vertices adjacent to v and $N_H[v] = N_H(v) \cup \{v\}$. For a vertex set X, we use $N_H(X) = \bigcup_{v \in X} N_H(v) \setminus X$. For u, v in graph H, let $d_H(u, v)$ denote the distance between u and v in H. For a graph H, we use $D(H) = max_{u,v \in H} d_H(u, v)$ to denote the diameter of H.

2 Inapproximability results

In this section, we show that unless $P = NP$, for any $\epsilon > 0$, there is no $(\frac{4}{3} - \epsilon)$-approximation algorithm running in polynomial time for the SINGLE ALLOCATION k-HUB CENTER problem.

Lemma 1. *For any $\epsilon > 0$, it is NP-hard to approximate the* SINGLE ALLOCATION k-HUB CENTER *problem to a ratio $\frac{4}{3} - \epsilon$.*

Proof. We reduce the SINGLE ALLOCATION k-HUB CENTER problem to the following SET COVER problem.

SET COVER
Input: A universe $\mathcal{U}$ of elements, $|\mathcal{U}| = n$ and a collection $\mathcal{S}$ of subsets of $\mathcal{U}$.
Output: A subset $\mathcal{S}^* \subseteq \mathcal{S}$ of minimum cardinality such that $\bigcup_{s_i \in \mathcal{S}^*} s_i = \mathcal{U}$.

Let $(\mathcal{U}, \mathcal{S})$ be an input instance of SET COVER. We construct a metric graph $G = (V = \mathcal{U} \cup \mathcal{S} \cup \{p\}, E, w)$ of the SINGLE ALLOCATION k-HUB CENTER problem according to $(\mathcal{U}, \mathcal{S})$. We define the cost of edges as follows.

- For $u, v \in \mathcal{U}$, $w(u, v) = 2$.
- For $v \in \mathcal{U}$ and $s \in \mathcal{S}$, if $v \in s$, $w(v, s) = 1$; otherwise $w(v, s) = 2$.
- For $s_i, s_j \in \mathcal{S}$, $w(s_i, s_j) = 1$.
- For $s \in \mathcal{S}$, $w(p, s) = 2$.
- For $v \in \mathcal{U}$, $w(p, v) = 3$.

It is not hard to see that G is a metric graph. Let H^* be an optimal solution of SINGLE ALLOCATION k-HUB CENTER problem in G and C^* be the set of k hubs in H^*. Suppose that $\mathcal{S}^* \subset \mathcal{S}$ is an optimal solution of SET COVER satisfying that $1 < |\mathcal{S}^*| = k' = k - 1 < n$. We now construct a spanning subgraph H by choosing vertices in $\mathcal{S}^*$ and p as hubs, *i.e.,* $C = \mathcal{S}^* \cup \{p\}$. For each $v \in \mathcal{U}$, connect v to a set $s \in \mathcal{S}^*$ such that $v \in s$ and $w(v, s) = 1$. Notice that such set s must exist since $\mathcal{S}^*$ is a set cover. Connect all vertices in $\mathcal{S} \setminus \mathcal{S}^*$ to a vertex in $\mathcal{S}^*$. We see that $D(H) = 3$ and $D(H^*) \leq 3$.

Notice that for $v \in \mathcal{U}$, $d_{H^*}(p, v) \geq w(p, v) = 3$. Thus, $D(H^*) \geq 3$. Since $D(H^*) \leq 3$ and $D(H^*) \geq 3$, we have $D(H^*) = 3$.

Suppose that there exists an approximation algorithm that finds a solution H of SINGLE ALLOCATION k-HUB CENTER problem in G and $D(H) < 4$. Let C be the set of the chosen k hubs in H. Let $\mathcal{U}' = \mathcal{U} \cap C$ and $\mathcal{S}' = \mathcal{S} \cap C$.

CLAIM 1. $p \in C$.

PROOF OF CLAIM. Suppose that $p \notin C$. Let $f(p) \in C$ be the unique neighbor of p in C. If $f(p) \in \mathcal{U}'$, then for $v \in V \setminus (C \cup \{p, f(p)\}$, $d_H(p, v) \geq 4$, a contradiction to the assumption that $D(H) < 4$. If $f(p) \in \mathcal{S}'$, then all vertices in $V \setminus C$ must be adjacent to $f(p)$; otherwise $D(H) \geq 4$. Moreover, for each $v \in \mathcal{U}$, $w(v, f(p)) = 1$. Thus $\{f(p)\}$ forms a set cover of $(\mathcal{U}, \mathcal{S})$. This contradicts to the assumption that the optimal solution of SET COVER is of size $k' > 1$. Hence, $p \in C$. ∎

CLAIM 2. $N_H(p) \setminus C = \emptyset$.

PROOF OF CLAIM. Suppose that $N_H(p) \setminus C \neq \emptyset$. Let $v \in N_H(p) \setminus C$. If $v \in \mathcal{U}$, we see that for $u \in \mathcal{U} \setminus \{v\}$,

$$\begin{aligned} d_H(u,v) &= w(v,p) + d_H(u,p) \\ &= 3 + d_H(u,p) \geq 6. \end{aligned}$$

If $v \in \mathcal{S}$, we see that for $u \in \mathcal{U}$,

$$\begin{aligned} d_H(u,v) &= w(v,p) + d_H(u,p) \\ &= 2 + d_H(u,p) \geq 5. \end{aligned}$$

Both cases are contradicted to the assumption that $D(H) < 4$. This shows that $N_H(p) \setminus C = \emptyset$. ∎

CLAIM 3. $N_H(\mathcal{U}') \setminus C = \emptyset$.

PROOF OF CLAIM. By Claim 1, $p \in C$. If there exists $v \in N_H(\mathcal{U}') \setminus C$, then $d_H(p, v) \geq 4$, a contradiction to the assumption that $D(H) < 4$. Thus, $N_H(\mathcal{U}') \setminus C = \emptyset$. ∎

CLAIM 4. *If $D(H) < 4$, then there exists a set cover of $(\mathcal{U}, \mathcal{S})$ of size at most k' where $k' = k - 1$.*

PROOF OF CLAIM. By Claims 1–3, we see that $p \in C$, $N_H(p) \setminus C = \emptyset$, and $N_H(\mathcal{U}') \setminus C = \emptyset$. If $D(H) < 4$, then for $v \in V \setminus C$, $w(v, f(v)) = 1$. Thus, for $v \in \mathcal{U} \setminus \mathcal{U}'$, $f(v)$ must be a set in $\mathcal{S}'$ such that $v \in f(v)$. We have $\mathcal{S}'$ is a set cover of $\mathcal{U} \setminus \mathcal{U}'$. For each $u \in \mathcal{U}'$, we pick exactly one set $s \in \mathcal{S}$ satisfying $u \in s$ in $\mathcal{S}''$. Note that $|\mathcal{S}'| + |\mathcal{U}'| = k - 1$ and $|\mathcal{U}'| = |\mathcal{S}''|$. We obtain a set cover $\mathcal{S}' \cup \mathcal{S}''$ of size at most $k' = k - 1$. This shows that if $D(H) < 4$, there exists a set cover of $(\mathcal{U}, \mathcal{S})$ of size at most k' where $k' = k - 1$. ∎

By Claims 1–4, if a solution H of SINGLE ALLOCATION k-HUB CENTER problem of diameter $D(H) < 4$ can be found in polynomial time, then SET COVER can be solved in polynomial time. However, SET COVER is a well-known NP-hard problem in the literature [5]. Therefore, for any $\epsilon > 0$, to approximate the SINGLE ALLOCATION k-HUB CENTER problem to a ratio $\frac{4}{3} - \epsilon$ is NP-hard. □

3 A 2-approximation algorithm

In this section, we give a 2-approximation algorithm for the SINGLE ALLOCATION k-HUB CENTER problem.

Algorithm BasicAPX$_{SAkHC}$

Let $U := V$. Initially, $C = \emptyset$. Construct a spanning subgraph H of G by the following steps.

Step 1: Pick k vertices $\{v_1, v_2, \ldots, v_k\}$ in U. Let $C := C \cup \{v_1, v_2, \ldots, v_k\}$ and $U := U \setminus \{v_1, v_2, \ldots, v_k\}$.

Step 2: Connect all vertices in U to v_1.

Step 3: Return H.

Theorem 1. *There is a 2-approximation algorithm for the* SINGLE ALLOCATION k-HUB CENTER *problem running in time $O(n)$.*

Proof. It is easy to see that in time $O(n)$ Algorithm BasicAPX$_{SAkHC}$ returns a spanning subgraph of G satisfying that C is a clique of size k in H; $V \setminus C$ forms an independent set in H; and each vertex in $V \setminus C$ is adjacent to exactly one vertex in C.

We now show that H is a 2-approximate solution. Let H^* denote an optimal solution of the SINGLE ALLOCATION k-HUB CENTER problem and $D(H^*)$ is the diameter of H^*. For $v \in V \setminus C$, we use $f(v)$ to denote its unique neighbor in C.

Note that for $u, v \in V$, $w(u, v) \leq D(H^*)$. We have the following cases.

- For $u, v \in C$, we see that

$$d_H(u,v) = w(u,v) \leq D(H^*).$$

- For $u \in V \setminus C$ and $v \in C$, we see that

$$\begin{aligned} d_H(u,v) &= w(u,v_1) + w(v_1,v) \\ &\leq 2 \cdot D(H^*). \end{aligned}$$

- For $u, v \in V \setminus C$, we see that

$$\begin{aligned} d_H(u,v) &= w(u,v_1) + w(v,v_1) \\ &\leq 2 \cdot D(H^*). \end{aligned}$$

Thus, for $u, v \in V$,

$$d_H(u,v) \leq 2 \cdot D(H^*).$$

This completes the proof. □

4 A 5/3-approximation algorithm

In this section, we give a 5/3-approximation algorithm for SINGLE ALLOCATION k-HUB CENTER problem.

Let H^* be an optimal solution of SINGLE ALLOCATION k-HUB CENTER problem in G. We use C^* to denote the size-k clique in H^*. For each $v \in V \setminus C^*$, let $f^*(v)$ denote its unique neighbor in C^* in H^*, *i.e.,* $N_{H^*}(v) = \{f^*(v)\}$. Let $z = \arg\max_{v \in V \setminus C^*}\{w(v, f^*(v))\}$ and $\ell = w(z, f^*(z))$.

Algorithm APX$_{SAkHC}$

Step 1: Run Algorithm A.

Step 2: Run Algorithm B.

Step 3: Return a best solution found by Algorithm A and Algorithm B.

Algorithm A

For $y, z \in V$, let $\ell = w(y, z)$, $U = V \setminus \{y, z\}$, and $C = \{y\}$. Construct a spanning subgraph H of G by the following steps and return a best solution with minimum diameter.

Step 1: Let $c_1 = y$ and connect z to c_1 in H.

Step 2: For each $v \in U$, if $w(v, c_1) \leq \ell$, connect v to c_1, and $U := U \setminus \{v\}$.

Step 3: While $i = |C| + 1 \leq k$ and $U \neq \emptyset$, do the following steps:

- choose $v \in U$, let $c_i = v$, and let $U := U \setminus \{v\}$ and let $C := C \cup \{c_i\}$;
- for $u \in U$, if $w(u, c_i) \leq 2\ell$, then connect u to c_i in H and $U := U \setminus \{u\}$.

Step 4: If $|C| < k$ and $U = \emptyset$, select $k - |C|$ vertices closest to y from $V \setminus C$ to be hubs, call the new spanning subgraph H'; otherwise let $H' := H$.

Algorithm B

For $y, z \in V$, let $\ell := w(y, z)$ and $C := \{y\}$. Construct a spanning subgraph H'' of G by the following steps and return a best solution with minimum diameter.

Step 1: Pick $k-1$ vertices $\{v_1, v_2, \ldots, v_{k-1}\}$ from $V \setminus \{y, z\}$ that are closest to y. Let $C := C \cup \{v_1, v_2, \ldots v_{k-1}\}$.

Step 2: Connect y to each vertex in $V \setminus C$.

Lemma 2. *Algorithm A finds a* $(1 + 4\delta)$-*approximation solution of* SINGLE ALLOCATION k-HUB CENTER *problem in time* $O(kn^3)$ *where* $\delta = \frac{\ell}{D(H^*)}$.

Proof. Let H^* be an optimal solution of SINGLE ALLOCATION k-HUB CENTER problem in G. We use C^* to denote the size-k clique in H^*. For each $v \in V \setminus C^*$, let $f^*(v)$ denote its unique neighbor in C^* in H^*, *i.e.,* $N_{H^*}(v) = \{f^*(v)\}$. Let $z = \arg\max_{v \in V \setminus C^*}\{w(v, f^*(v))\}$ and $\ell = w(z, f^*(z))$. Let $c_1 = f^*(z)$. Suppose the algorithm guesses correct y, z and $w(y, z) = \ell$.

In H^*, if we remove edges with both end vertices in $C^* = \{s_1, s_2, \ldots, s_k\}$, then the remaining graph consists of k components and each component is a star. Let $S_1, S_2, \ldots, S_k$ be the k stars. Note that s_i is the center of star S_i for $i = 1, 2, \ldots, k$. W.l.o.g., assume that $c_1 = s_1$. Since for each $v \in V \setminus C^*$, $w(v, f^*(v)) \leq \ell$, for $u, v \in S_i$, $d_{H^*}(u, v) \leq 2\ell$. Since the algorithm adds edges (v, c_1) in H if $w(v, c_1) \leq \ell$, we see that $S_1 \subset N_H[c_1] \setminus C$. Notice that for each S_j, $j \geq 2$, if there exists $v \in S_j$ specified as $c_i \in C$, then all the other vertices in S_j are connected to one of $c_1, c_2, \ldots, c_i$ in H. Moreover, for each c_i, $1 < i \leq |C|$, there exists S_j, $1 < j \leq k$, such that $c_i \in S_j$ and $S_j \cap C = \{c_i\}$. Notice that if there exists S_j, $1 < j \leq k$, $S_j \cap C = \emptyset$, then all vertices of S_j must be connected to one of vertices in C in H and $|C| < k$. We see that if $|C| = k$, then H is a feasible solution. Suppose that $|C| < k$ and we select $k - |C|$ vertices closest to y from $V \setminus C$ to be hubs. Call the new spanning subgraph H' and let C' be the set of new hubs. Notice that $|C \cup C'| = k$, $C \cap C' = \emptyset$, and vertices in C' are not adjacent to any vertex in $V \setminus (C \cup C')$. We see that H' is a feasible solution.

Now we show that $D(H') \leq D(H^*) + 4\ell$. There are three cases.

Case 1. For $u, v \in C \cup C'$,

$$d_{H'}(u, v) = w(u, v) \leq d_{H^*}(u, v) \leq D(H^*).$$

Case 2. For $u \in V \setminus (C \cup C')$ and $v \in C \cup C'$,

$$\begin{aligned} d_{H'}(u, v) &= w(u, f(u)) + w(f(u), v) \\ &\leq D(H^*) + 2\ell \end{aligned}$$

where $f(u)$ denotes the unique neighbor of u in C in both H and H'.

Case 3. For $u, v \in V \setminus (C \cup C')$,

$$\begin{aligned} d_{H'}(u,v) &= d_H(u,v) \\ &= w(u,f(u)) + w(f(u),f(v)) \\ &\quad +w(v,f(v)) \\ &\leq D(H^*) + 4\ell \end{aligned}$$

where $f(u), f(v) \in C$ denote the neighbors of u and v in both H and H', respectively.

Thus, $D(H') \leq D(H^*) + 4\ell$. We obtain that

$$\frac{D(H')}{D(H^*)} \leq \frac{D(H^*) + 4\ell}{D(H^*)} = 1 + 4\delta$$

where $\delta = \frac{\ell}{D(H^*)}$.

The algorithm guess y, z such that $y = s_1$ and $w(y,z) = \ell$. There are $O(n^2)$ possibilities of the pair $\{y, z\}$. In Algorithm A, there are $O(n^2)$ spanning subgraphs are constructed. It is not hard to see that it takes $O(kn)$ to construct a spanning subgraph H'. Thus, the running time of Algorithm A is $O(kn^3)$. This completes the proof. □

Lemma 3. *Algorithm B finds either an optimal solution of the* Single Allocation k-Hub Center *problem or a* $(2-2\delta)$*-approximation solution of* Single Allocation k-Hub Center *problem in time* $O(kn^3)$ *where* $\delta = \frac{\ell}{D(H^*)} < 1/2$.

Proof. Suppose that H^* is an optimal solution. Let (y,z) be a longest edge among all edges with one end vertex in C^* and the other end vertex in $V \setminus C^*$ and $w(y,z) = \ell$ where $y \in C^*$ and $z \in V \setminus C^*$. Let H'' be the solution returned by Algorithm B.

For $v \in V \setminus \{z\}$,

$$\begin{aligned} d_{H''}(v,y) &= w(v,y) \\ &\leq d_{H^*}(v,z) - w(y,z) \\ &\leq D(H^*) - \ell. \end{aligned}$$

This shows that for $v \in V \setminus \{z\}$,

$$d_{H''}(v,y) \leq D(H^*) - \ell.$$

For $v \in V \setminus \{z\}$, we have

$$\begin{aligned} d_{H''}(z,v) &\leq w(z,y) + d_{H''}(y,v) \\ &\leq \ell + D(H^*) - \ell \\ &= D^*(H). \end{aligned}$$

For each $u, v \in V \setminus \{z\}$, we see that

$$\begin{aligned} d_{H''}(u,v) &= d_{H''}(u,y) + d_{H''}(v,y) \\ &\leq 2 \cdot D(H^*) - 2\ell. \end{aligned}$$

It is easy to see that $D(H'') \geq D(H^*)$. If $D(H^*) \leq 2\ell$, then

$$D(H'') \leq 2 \cdot D(H^*) - 2\ell \leq D(H^*).$$

Thus, if $D(H'') \leq 2\ell$, we obtain that

$$D(H'') = D(H^*).$$

Suppose that $D(H^*) > 2\ell$. We see that

$$\frac{D(H'')}{D(H^*)} \leq \frac{2 \cdot D(H^*) - 2\ell}{D(H^*)} = 2 - 2\delta$$

where $\delta = \frac{\ell}{D(H^*)} < \frac{1}{2}$.

Algorithm B guess y and z, there are $O(n^2)$ possibilities of y and z. In Algorithm B, there are $O(n^2)$ spanning subgraphs are constructed. It takes $O(kn)$ time to construct a spanning subgraph. Thus, the running time of Algorithm B is $O(kn^3)$. This completes the proof. □

Theorem 2. *There is a* $\frac{5}{3}$*-approximation algorithm for the* Single Allocation k-Hub Center *problem running in time* $O(kn^3)$ *where* n *is the number of vertices in the input graph.*

Proof. By Lemma 2, it takes $O(kn^3)$ time to find a $(1 + 4\delta)$-approximation solution for the Single Allocation k-Hub Center problem where $\delta = \frac{\ell}{D(H^*)}$. By Lemma 3, it takes $O(kn^3)$ time either to find an optimal solution or to find a $(2-2\delta)$-approximation solution for the same problem. In Step 3 of Algorithm APX_{SAkHC}, it takes $O(1)$ time to find a best solution returned by Algorithm A and Algorithm B. The worst approximation ratio happens when $1 + 4\delta = 2 - 2\delta$ and $\delta = 1/6$. Therefore, the approximation ratio is $5/3$ and the running time of Algorithm APX_{SAkHC} is $O(kn^3)$. This completes the proof. □

5 Concluding remarks

In this paper, we give a lower bound $\frac{4}{3} - \epsilon$ and upper bound $\frac{5}{3}$ of the approximability of the Single Allocation k-Hub Center problem. For the future work, it is interesting to see whether the gap between lower and upper bounds can be reduced. One possibility is to show that for any $\epsilon > 0$, it is NP-hard to approximate the Single Allocation k-Hub Center problem to a ratio $\alpha - \epsilon$ where $\alpha > \frac{4}{3}$. The other possibility is to design a γ-approximation algorithm for the Single Allocation k-Hub Center problem and $\gamma < \frac{5}{3}$.

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