# The Clique Transversal and Clique Independence of Distance Hereditary Graphs

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**Abstract**

*A clique-transversal set of a graph $G$ is a subset of vertices intersecting all maximal cliques of $G$. The smallest possible cardinality $\tau_C(G)$ among all clique transversal sets for $G$ is called clique transversal number. A clique independence set is a collection of vertex-disjoint maximal cliques of $G$. The largest possible cardinality $\alpha_C(G)$ among all clique independent sets for $G$ is called clique independence number. If equality of $\tau_C(H)$ and $\alpha_C(H)$ always holds for every induced subgraph $H$ of $G$, then the graph is called clique-perfect.*

*The paper concentrates on the clique transversal and clique independence of distance-hereditary graphs. We present two polynomial time algorithms for finding the clique transversal number and the clique independence number of a given distance-hereditary graph.*

## 1 Introduction

The concept of clique transversal has been mentioned by Payan [18] in 1979. Formally, the clique transversal problem (CTP) can be described as follows. Consider a simple graph $G = (V, E)$ without multi-edges and selfloops. A *clique* is a subset of pairwise adjacent vertices of $G$. A *maximal clique* is a clique that is not a proper subset of any other clique. A *clique-transversal set* of $G$ is a subset of vertices intersecting all maximal cliques of $G$. The smallest possible cardinality $\tau_C(G)$ among all clique-transversal sets is called *clique-transversal number*. *The clique transversal problem* (CTP) is finding a minimum cardinality clique transversal set of $G$.
There are by now numerous algorithms known for clique transversal problem (CTP). In 1990, Tuza initiated [19] the study of CTP for family of chordal graphs, and then several theoretical results for the clique-transversal number of chordal graphs were presented in [19, 1, 2, 3, 13]. Furthermore, it was shown [12] that the problem of deciding whether a chordal graph has two disjoint minimum-cardinality clique transversal sets is NP-complete. In regard to the complexity of CTP, solving the problem for graphs in general is NP-hard even though the problem is restricted to co-comparability graphs, planar graphs, line graphs, total graphs, split graphs, undirected path graphs, and $k$-tree with unbounded $k$ (see [8, 9, 16]). It is unlikely that there exist polynomial-time algorithms to solve NP-hardness problems. Nevertheless, for comparability graphs, strongly chordal graphs, helly circular-arc graphs the CTP is polynomially solvable [4, 8, 9, 16]. Certainly, the problem is polynomially solvable for subclasses of the graph classes as well.

We show another interesting problem, *clique-independence problem*. A *clique independent set* is a collection of vertex-disjoint maximal cliques of a graph $G$. The largest possible cardinality $\alpha_C(G)$ among all clique-independent sets of $G$ is called *clique-independence number*. The *clique independence problem* is finding a maximum cardinality clique-independent set of $G$. Apparently, the clique-transversal number is at least as large as the clique independence number. If equality of $\tau_C(H)$ and $\alpha_C(H)$ always holds for every induced subgraph $H$ of $G$, then the graph is called *clique-perfect*. Durán [7] et al. showed that parameters $\tau_C(G)$ and $\alpha_C(G)$ can be computed in polynomial time for graphs belonging to the class $\mathcal{G}$. Their method employed integer linear programming and also led to the conclusion the $\mathcal{G}$ is a clique perfect graph class. M.S. Chang et al. [9] gave an NP-hardness proof for the CIP restricted to $k$-tree with unbounded $k$. In [8], the authors gave a linear-time algorithm to solve CIP on strongly chordal graphs and proved the clique-perfectness of strongly chordal graphs. Comparability graphs are also clique-perfect. The soundness of that has been shown in [4] and the authors presented an $O(m\sqrt{n}+M(n))$ time algorithm to compute the clique-independence number, where $M(n)$ is the complexity of multiplying two $n\times n$ matrices. For each helly circular-are graph $G$ an $O(n^2)$ time algorithm was given by [16] to solve CIP and $\alpha_C(G)\le\tau_C(G)\le\alpha_C(G)+1$.

Given a graph $G$ with a weight function $w: V\to R^+$. When $w$ is the constant function $\mathbf{1}$, a minimum weighted clique transversal set of $G$ is also a minimum cardinality clique transversal set of unweighted $G$.

In the paper, we exhibit polynomial time algorithms to find a minimum weighted clique transversal set and compute the clique independence number of a given distance-hereditary graph.

## 2 Preliminaries

We use mainly standard terminology and notation on undirected graphs in [15]. All considered graphs in the paper are undirected, weighted, and simple without multi-edges and selfloops. To avoid tautology, please see the definitions of clique transversal and clique independence of a graphs in prior section. Assume $G=(V,E)$ be a graph. Denote $|V|$=$n$ and $|E|$=$m$. For a subset $V'$ of $V$, we denote $G[V']$ to be the subgraph induced by $V'$. Let $S$ and $X$ be two sets of weighted sets. $\max S$ and $\min S$ represent a maximum and a minimum weighted set of $S$, respectively. Let $S+X$ stand for $S\cup X$ and $S-X$ stand for $S\setminus X$. Let $I$ be a set of integers. $\max I$ (resp. $\min I$) represents the maximum (resp. minimum) integer of $I$.

Recall the definition of a distance-hereditary graph. A graph $G=(V,E)$ is *distance hereditary* if each pair of vertices are equidistant in every connected induced subgraph containing them. The family of distance-hereditary graphs has a graceful characterization, appeared in [10]. The characterization makes use of so-called twin sets. Every distance hereditary graph has a *twin set* that is a subset of vertices. We use $TS(G)$ to denote a twin set of a distance hereditary graph $G$ thereinafter. For references to other characterizations (such as [5, 17]) we refer to [6].

**Theorem 1** *Distance hereditary graphs can be defined recursively as follows:*

1. *A graph consisting of only one vertex is distance hereditary. The twin set is the vertex itself.*
2. *If $G_1$ and $G_2$ are distance hereditary, then the disjoint union $G$ of them is also distance hereditary. $TS(G)$=$TS(G_1)\cup TS(G_2)$. We call this the* **false twin** *operation.*
3. *If $G_1$ and $G_2$ are distance hereditary, then the graph $G$ obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$. That is , $TS(G_1)$ and $TS(G_2)$ form a join. $TS(G)$ is again the union of $TS(G_1)$ and $TS(G_2)$. We call this the* **true twin** *operation.*
4. *If $G_1$ and $G_2$ are distance hereditary, then the graph $G$ obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$. The twin set of $G$ is $TS(G_1)$. We call this the* **pendant vertex** *operation.*

By the theorem above, a binary ordered decomposition tree can easily be obtained in linear time. In the decomposition tree, each leaf is a single vertex graph and each node represents one of the three operations, pendant vertex operation (P), true twin operation (T), and false twin operation (F). The ordered decomposition tree is called **PTF**-tree.

For each of the operations in Theorem 1 we show thereinafter how to find a minimum weighted clique transversal set and compute the clique independence number.

The following lemma is a basic observation.

**Lemma 1** *Given a distance-hereditary graph $G$ with a twin set $TS(G)$. If $G$ is obtained by a true twin operation or by a pendant vertex operation of two distance-hereditary graphs $G_1$ and $G_2$, then a clique transversal set $S$ of $G$ must satisfy at least one statement of the following ones:*

1. *$S\cap TS(G_1)$ must be a clique transversal set of the graph induced by $TS(G_1)$.*
2. *$S\cap TS(G_2)$ must be a clique transversal set of the graph induced by $TS(G_2)$.*

**Proof.** Assume that a clique-transversal set $S$ does not satisfy any statements mentioned in Lemma 1. There must exist a maximal clique $c_1$ (resp. $c_2$) of the graph induced by $TS(G_1)$ (resp. $TS(G_2)$) such that $S$ and $c_1$ (resp. $c_2$) have no intersection. Since $G$ is obtained by a true operation or a pendant vertex operation of $G_1$ and $G_2$, $TS(G_1)$ and $TS(G_2)$ form a join. Similarly, $c_1$ and $c_2$ form a join, and the join of them is also a maximal clique of $G$. That $S$ cannot intersect the maximal clique is a contradiction. Hence $S$ must satisfy at least one statement of Lemma 1. □

## 3 Weighted Clique Transversal of Distance-Hereditary Graphs

In this section, it is our intention to find a minimum weighted clique-transversal set (**MWCTS**) of a given distance-hereditary graph with a weight function $w$. We maintain three essential sets, $MWCTS(G)$, $MT(G)$, and $NT(G)$ for each distance hereditary graph $G = (V, E)$ with twin set TS(G). These sets represent the following:

1. $MWCTS(G)$ is a **MWCTS** of $G$.
2. $MT(G)$ is a **MWCTS** of $G$ that intersects all maximal cliques of $G[TS(G)]$. Notice that a maximal clique of $G[TS(G)]$ may not be a maximal clique of $G$.
3. $NT(G)$ is a minimum weighted subset of $V$ that intersects all maximal cliques of $G$ except for the maximal cliques formed by the vertices of $TS(G)$.

In the following lemmas, we show how these sets can be computed for each of the operations defined in Theorem 1.

**Lemma 2** *If $G$ has exactly one vertex $v$, then*

1. $MWCTS(G) = \{v\}$.
2. $MT(G) = \{v\}$.
3. $NT(G) = \emptyset$.

**Proof.** These formulas are straightforward by definition. □

**Lemma 3** *Assume $G$ is obtained by a* **false twin** *operation of two distance hereditary graphs $G_1$ and $G_2$. Formulas are presented as follows:*

1. $MWCTS(G) = MWCTS(G_1) \cup MWCTS(G_2)$.
2. $MT(G) = MT(G_1) \cup MT(G_2)$.
3. $NT(G) = NT(G_1) \cup NT(G_2)$.

**Proof.** These formulas can be easily verified by definition. □

**Lemma 4** *Assume $G$ is obtained by a* **true twin** *operation of two distance hereditary graphs $G_1$ and $G_2$. Formulas are presented as follows:*

1. $MWCTS(G) = \min\{MT(G_1) \cup NT(G_2), MT(G_2) \cup NT(G_1)\}$.
2. $MT(G) = \min\{MT(G_1) \cup NT(G_2), MT(G_2) \cup NT(G_1)\}$.
3. $NT(G) = NT(G_1) \cup NT(G_2)$.

**Proof. Formula 3** is straightforward. Consider $MWCTS(G)$. Since $MWCTS(G)$ is a clique transversal set of $G$, by Lemma 1 $MWCTS(G)$ must intersect all maximal cliques of $G[TS(G)]$. Consider the following cases for $MWCTS(G)$. If $MWCTS(G) \cap TS(G_i)$ can intersect all maximal cliques of $G[TS(G)]$ but $MWCTS(G) \cap TS(G_{3-i})$ cannot , then $MWCTS(G) = MT(G_i) \cup NT(G_{3-i})$ $(i = 1, 2)$. If both of $MWCTS(G) \cap TS(G_1)$ and $MWCTS(G) \cap TS(G_2)$ can intersect all maximal cliques of $G[TS(G)]$. Then $MWCTS(G) = MT(G_1) \cup MT(G_2)$. However, we claim that in this case the total weight $W_1$ of $MT(G_1)$ equals the total weight $W_1'$ of $NT(G_1)$ and the total weight $W_2$ of $MT(G_2)$ equals the total weight $W_2'$ of $NT(G_2)$.

(1) Let $W$ be the total weight of $MWCTS(G)$ and now $W = W_1 + W_2$.

(2) $W_1 \geq W_1'$ and $W_2 \geq W_2'$. Hence, $W_1 + W_2 \geq W_1 + W_2'$.

(3) $MT(G_1) \cup NT(G_2)$ is a clique transversal set of $G$. Hence, $W_1 + W_2 = W \leq W_1 + W_2'$.

$W_2 = W_2'$ by standing on (2) and (3). Similarly, $W_1 = W_1'$. Thus $W = W_1 + W_2' = W_1' + W_2$. That is to say, for finding a $MWCTS(G)$ in this case, we can devote our attention to finding a $MT(G_1) \cup NT(G_2)$ and finding a $MT(G_2) \cup NT(G_1)$.
$MT(G)$ is also a clique transversal set of $G$. By Lemma 1, $MT(G)$ must intersect all maximal cliques of $G[TS(G)]$. Hence, by using the same argument as the proof of $MWCTS(G)$, **Formula 2** is true. □

**Lemma 5** *Assume $G$ is obtained by a* **pendant vertex** *operation $G_2$ to $G_1$. Formulas are presented as follows:*

1. $MWCTS(G) = \min\{MT(G_1) \cup NT(G_2), MT(G_2) \cup NT(G_1)\}$.
2. $MT(G) = MT(G_1) \cup NT(G_2)$.
3. $NT(G) = \min\{MT(G_1) \cup NT(G_2), MT(G_2) \cup NT(G_1)\}$.

**Proof Formula 2** is easily seeing by definition. Consider $MWCTS(G)$. Since $MWCTS(G)$ is a clique transversal set of $G$, by Lemma 1 $MWCTS(G)$ must intersect all maximal cliques of $G[TS(G)]$. By using the same argument as the proof of Lemma 4, **Formula 1** holds.
Consider $NT(G)$. $G$ is obtained by a pendant vertex operation of $G_1$ and $G_2$. All maximal cliques of $G[TS(G)]$ and all maximal cliques of $G[TS(G)]$ form new maximal cliques for $G$ by the join of $TS(G_1)$ and $TS(G_2)$. However $TS(G) = TS(G_1)$ now. $NT(G)$ must intersect these new maximal cliques. Consider the following cases:

1. If $N(G) \cap TS(G_i)$ can but $N(G) \cap TS(G_{3-i})$ cannot intersect these new maximal cliques $(i = 1, 2)$.
2. If both of $N(G) \cap TS(G_1)$ and $N(G) \cap TS(G_2)$ can intersect these new maximal cliques.

Clearly, the formula of $NT(G)$ can be proved true by using the similar way to the proof of Lemma 4. □

**Theorem 2** *Given a distance-hereditary graph with a* **PTF**-*tree. A minimum weighted clique transversal set of distance-hereditary graph can be obtained in linear time.*

## 4 The Clique Independence of Distance Hereditary Graphs

The section is concerned with the *clique independence number* $\alpha_C(G)$ of a given distance-hereditary graph $G = (V, E)$ with a twin set $TS(G)$. Assume the size of $V$ is $n$. For further convenience, we have the following definitions:

1. $\alpha_C(G)$ is the *clique independence number* of $G$.
2. For $0 \le i \le n$, $\alpha_i(G)$ is the maximum cardinality among all the clique independent sets of $G$ which contains exactly $i$ vertex-disjoint maximal cliques formed by the vertices of $TS(G)$ .
3. For $0 \le i \le n$, $X_i(G)$ is a clique independent set of $G[TS(G)]$ with size $i$ and $Y(G)$ is a clique independent set of $G$ without any vertex-disjoint maximal cliques which are formed by the vertices of $TS(G)$. For every $y \in Y(G)$ and $x \in X_i(G)$, $x \cap y = \emptyset$.
4. For $0 \le i \le n$, $\overline{\alpha_i}(G)$ is the maximum cardinality among all such sets $X_i(G) \cup Y(G)$.

We maintain for each distance-hereditary graph $G$ with a twin set $TS(G)$ parameters $\alpha_C(G)$, $\alpha_i(G)$ and $\overline{\alpha_i}(G)$ for $1 \le i \le n$.

**Lemma 6** *If $G$ consists of only one vertex $v$, then*

1. $\alpha_C(G) = 1$.
2. *For* $0 \le i \le n$, $\alpha_i(G) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$
3. *For* $0 \le i \le n$, $\overline{\alpha_i}(G) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$

**Proof.** These formulas are straightforward by definition. □

**Lemma 7** *Assume $G$ is obtained by a* **false twin** *operation of $G_1$ and $G_2$. $TS(G)$=$TS(G_1) \cup TS(G_2)$. Formulas are presented as follows:*

1. $\alpha_C(G) = \alpha_C(G_1) + \alpha_C(G_2)$.
2. *For* $0 \le i \le n$, $\alpha_i(G) = \max\limits_{0 \le j \le n} \{A_j\}$, *where*
$$A_j = \begin{cases} 0 & \text{if } j > 0 \text{ and } \alpha_j(G_1) = 0 \text{ or} \\ & i - j > 0 \text{ and } \alpha_{i-j}(G_2) = 0 \\ & \\ \alpha_j(G_1) + \alpha_{i-j}(G_2) & \text{otherwise.} \end{cases}$$
3. *For* $0 \le i \le n$, $\overline{\alpha_i}(G) = \max\limits_{0 \le j \le n} \{\overline{A_j}\}$, *where*
$$\overline{A_j} = \begin{cases} 0 & \text{if } j > 0 \text{ and } \overline{\alpha_j}(G_1) = 0 \text{ or} \\ & i - j > 0 \text{ and } \overline{\alpha_{i-j}}(G_2) = 0 \\ & \\ \overline{\alpha_j}(G_1) + \overline{\alpha_{i-j}}(G_2) & \text{otherwise.} \end{cases}$$

**Proof.** These formula can be easily verified by definition. □

**Lemma 8** *Assume $G$ is obtained by a* **true twin** *operation of $G_1$ and $G_2$. $TS(G) = TS(G_1) \cup TS(G_2)$. Formulas are presented as follows:*

1. $\alpha_C(G) = \max\limits_{0 \le i \le n} \{\alpha_i(G)\}$.
2. *For* $0 \le i \le n$,
$$\alpha_i(G) = \begin{cases} \overline{\alpha_i}(G_1) + \overline{\alpha_i}(G_2) - i \\ \text{if } \overline{\alpha_i}(G_1), \overline{\alpha_i}(G_2) > 0, \text{ or } i = 0 \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

*3.* *For* $0 \leq i \leq n$*,*

$$\overline{\alpha_i}(G) = \begin{cases} \overline{\alpha_i}(G_1) + \overline{\alpha_i}(G_2) - i \\ \text{if } \overline{\alpha_i}(G_1),\ \overline{\alpha_i}(G_2) > 0, \text{ or } i = 0 \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

**Proof. Formula 1** is obvious by definition. Consider **Formula 3**. $G$ is now obtained by a true twin operation of $G_1$ and $G_2$. $TS(G_1)$ and $TS(G_2)$ form a join, and $TS(G)$=$TS(G_1) \cup TS(G_2)$. By definition, a clique independent set, $Y(G)$ does not have any maximal cliques formed by vertices of $TS(G)$. Therefore, $Y(G)$ is a disjoint union of a clique independent set $Y(G_1)$ and a clique independent set $Y(G_2)$. Since $TS(G)$=$TS(G_1) \cup TS(G_2)$, every maximal clique of $G[TS(G_1)]$ and every maximal clique of $G[TS(G_1)]$ form a join, and the join is a maximal clique of $G[TS(G)]$. Similarly, the statement also reversely holds. To compute $\overline{\alpha_i}(G)$, we consider the cases below:
**Case 1.** $i = 0$.
All such sets $X_0(G)$ are empty sets. Therefore, $\overline{\alpha_0}(G)$ is the maximum cardinality among all such set $Y(G)$=$Y(G_1) \cup Y(G_2)$. Hence, $\overline{\alpha_0}(G)$=$\overline{\alpha_0}(G_1)$+ $\overline{\alpha_0}(G_2) - i$, where $i = 0$.
**Case 2.** $1 \leq i \leq n$.
By discussion above, every maximal clique of a clique independent set $X_i(G)$ is formed by a join of a maximal clique of $G[TS(G_1)]$ and a maximal clique of $G[TS(G_2)]$. Assume $X_i(G)$ consists of $i$ vertex-disjoint maximal cliques $S_1, \ldots, S_i$ and for $1 \leq k \leq i$, $S_k$ is a join of $c_k$ and $c'_k$, where $c_k$ is a maximal clique of $G[TS(G_1)]$ and $c'_k$ is a maximal clique of $G[TS(G_2)]$. Apparently, the set $\{c_1, \ldots, c_i\}$ is an $X_i(G_1)$, and the set $\{c'_1, \ldots, c'_i\}$ is an $X_i(G_2)$. We can have $|X_i(G)|=$ $|X_i(G_1)|$+$|X_i(G_2)| - i$. By definition, an $X_i(G)$ and a $Y(G)$ are disjoint , so that $|X_i(G) \cup Y(G)|$=$|X_i(G)|$+$|Y(G)|$. Hence $|X_i(G) \cup Y(G)|=$ $|X_i(G)|$+$|Y(G)|$=$(|X_i(G_1)| + |Y(G_1)|)$+$(|X_i(G_2)| + |Y(G_2)|) - i = (|X_i(G_1) \cup Y(G_1)|)$+$(|X_i(G_2) \cup Y(G_2)|) - i$. Consequently, $\overline{\alpha_i}(G) = \overline{\alpha_i}(G_1)$+ $\overline{\alpha_i}(G_2) - i$. However, if $\overline{\alpha_i}(G_1) = 0$ or $\overline{}(G_2) = 0$, this implies that all such sets $X_i(G_1) \cup Y(G_1) = \emptyset$ or $X_i(G_2) \cup Y(G_2) = \emptyset$. In other words, we can not obtain such sets $X_i(G) \cup Y(G)$. In this situation, $\overline{\alpha_i}(G) = 0$.

Consider **Formula 2**. We know that $TS(G) = TS(G_1) \cup TS(G_2)$. A maximal clique of $G$ formed by the vertices of $TS(G)$, is also a maximal clique of $G[TS(G)]$. The accuracy of **Formula 2** can be proved by the same arguments as **Formula 3** uses. □

**Lemma 9** *Assume* $G$ *is obtained by a* **pendant vertex** *operation of* $G_1$ *and* $G_2$*.* $TS(G) = TS(G_1)$*. Formulas are presented as follows:*

*1.* $\alpha_C(G) = \alpha_0(G)$.

*2.* *For* $1 \leq i \leq n$, $\alpha_i(G) = 0$.

*3.* $\alpha_0(G) = \max\limits_{0 \leq j \leq n} \{A_j\}$, *where*

$$A_j = \begin{cases} \overline{\alpha_j}(G_1) + \overline{\alpha_j}(G_2) - j \\ \text{if } \overline{\alpha_j}(G_1),\ \overline{\alpha_j}(G_2) > 0, \text{ or } j = 0 \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

*4.* *For* $0 \leq i \leq n$, $\overline{\alpha_i}(G) = \overline{\alpha_i}(G_1) \cup \overline{\alpha_0}(G_2)$.

**Proof.** $G$ is now obtained by a pendant vertex operation. $TS(G_1)$ and $TS(G_2)$ form a join, and $TS(G)$=$TS(G_1)$. A maximal clique, formed by the vertices of $TS(G)$, is not a maximal clique of $G$, so that $\alpha_C(G)$=$\alpha_0(G)$ and for $1 \leq i \leq n$, $\alpha_i(G)$=0. Assume that $S$ is a maximum cardinality clique independent set of $G$. To compute $\alpha_0(G)$, we consider the cases below:

1. $S$ doesn't contain any maximal cliques of $G$ obtained from the join of $TS(G_1)$ and $TS(G_2)$.
   Therefore, $S$ is a clique independent set of all such sets $Y(G) = Y(G_1) \cup Y(G_2)$. Then, $\alpha_0(G)$=$\overline{\alpha_0}(G_1)$+ $\overline{\alpha_0}(G_2) - j$, where $j = 0$.

2. For $1 \leq j \leq n$, $S$ contains exactly $j$ maximal cliques $S_1, \ldots, S_j$ of $G$ obtained from the join of $TS(G_1)$ and $TS(G_2)$.
   Let $S_{tw} = \{S_1, \ldots, S_j\}$. Since every maximal clique of $S_{tw}$ is formed by a join of a maximal clique of $G[TS(G_1)]$ and a maximal clique of $G[TS(G_2)]$. Assume that for $1 \leq k \leq j$, $S_k$ is a join of $c_k$ and $c'_k$, where $c_k$ is a maximal clique of $G[TS(G_1)]$ and $c'_k$ is a maximal clique of $G[TS(G_2)]$. Apparently, the set $\{c_1, \ldots, c_j\}$ is an $X_j(G_1)$, and the set $\{c'_1, \ldots, c'_j\}$ is an $X_j(G_2)$. In remainder of the proof for the case, we can use the similar arguments as **Lemma** 8 uses for its **Formula 3** to prove that **Formula 2** is true.

Consider **Formula 3**.
Since $TS(G)$=$TS(G_1)$, $X_i(G)$=$X_i(G_1)$, for $0 \leq i \leq n$. A set $X_i \cup Y(G)$ does not contain any maximal cliques of $G[TS(G_2)]$. We have that $|X_i(G) \cup Y(G)| = |X_i(G)|$+$|Y(G)| = |X_i(G_1)| + |Y(G_1)| + |Y(G_2)|$=$(|X_i(G_1)|$+$|Y(G_1)|)$+$(|X_0(G_2)|$+$|Y(G_2)|$ =$|X_i(G_1) \cup Y(G_1)|$+$|X_0(G_2) \cup Y(G_2)|$. Hence, $\overline{\alpha_i}(G_1)$ =$\overline{\alpha_i}(G_1)$+$\overline{\alpha_0}(G_2)$, for $0 \leq i \leq n$. □
We finally obtain the following result:

**Theorem 3** *Given a distance-hereditary graph with a* **PTF***-tree.* $\alpha_C(G)$ *can be obtained in* $O(n^3)$ *time.*

## 5 Conclusion

The paper concentrates on the clique transversal and clique independence of distance-hereditary graphs. We have shown how to find the clique transversal number and the clique independence number of a given distance-hereditary graph in polynomial time. However, we have not found any good method to verify the clique perfect of distance-hereditary graphs.

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