Fault Hamiltonicity of the Shuffle-Cubes *

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Abstract

Hypercubes are famous interconnection networks. The shuffle cubes are proposed as its variation with lower diameter. In this paper, we concentrate on the fault tolerant hamiltonicity and hamiltonian connectivity of the shuffle cube. We say that a graph G is f-hamiltonian if there is a hamiltonian cycle in $G-F$ and is f-hamiltonian connected if there is a hamiltonian path between any two vertices of $G-F$ for any $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$. We show that the n-dimensional shuffle cube is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected.

1 Introduction

The interconnection network have been an important research area for parallel and distributed computer systems. Designing an interconnection network is multi-objected and complicated. For simplifying this task, we usually use a graph to represent the network's topology where vertices represent processors and edges represent links between processors. The hypercube [14] is an example. The hypercube possesses many good properties and is implemented as many multi-processor systems. The n-dimensional hypercube, denoted by Q_n, is known that $D(Q_n) = n$ and $\kappa(Q_n) = n$ [14]. However, a hypercube does not make the best use of its hardware. It is possible to fashion networks with lower diameters than that of Q_n and with the same connectivity. For examples, the cross cubes [5, 6], twisted cubes [1, 7], and möbious cubes [4] are derived from Q_n by changing the connection of some hypercube links. All of these topologies have connectivity n and have diameter around $\frac{n}{2}$. Thus, this is an improvement of approximately a factor of 2.

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The shuffle cube was proposed by Li et al. [11] to reduce diameter more. The n-dimensional shuffle cube is denoted by SQ_n. SQ_n, which can be constructed recursively, is proved to be an n-regular graph with connectivity n. The diameter of SQ_n is $\lceil \frac{n}{4} \rceil + 3$. The wide diameter of SQ_n is bounded by $\lceil \frac{n}{4} \rceil + 7$ [12]. Yet, many other topological properties are not studied.

Hamiltonicity is an important property for network topologies. Thus, the existence of a hamiltonian cycle is a desired property for a new proposed topology. Hamiltonian connectivity is a related property of hamiltonicity, namely, there is a hamiltonian path between any two vertices of a graph. Since processors or links may be failed sometimes, fault-hamiltonicity and fault-hamiltonian connectivity are concerned in many studies on network topologies, such as hypercubes [3, 10], twisted cubes [9], deBruijn networks [13], and star graphs [15, 8]. We say that a graph G can tolerate f faults when embedding a hamiltonian cycle if there is a hamiltonian cycle in $G-F$ for any $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$. We use *f-hamiltonian* to denote this property of G. Similarly, we use *f-hamiltonian connected* to denote the property that there is a hamiltonian path between any two vertices in $G-F$ for any $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

In this paper, we show that the n-dimensional shuffle cube is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected. In Section 2, we introduce the definition of the shuffle cube and some properties. And then show the main result in Section 3.

2 Definition and some properties of the shuffle cubes

In this paper, we concentrate on loopless undirected graphs. For the graph definition and notation, we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices a and b are *adjacent* if $(a, b) \in E$. A *path* is represented by $\langle v_0, v_1, v_2 \cdots, v_k \rangle$. We also write the path $\langle v_0, v_1, v_2 \cdots, v_k \rangle$ as $\langle v_0, P_1, v_i, v_{i+1} \cdots, v_j, P_2, v_t, \cdots, v_k \rangle$, where P_1 is the path $\langle v_0, v_1, \cdots, v_i \rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \cdots, v_t \rangle$. A path is a *hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

We use n-bit binary string to represent a vertex. For example, vertex u can be labelled as $u_{n-1}u_{n-2} \cdots u_1u_0 \in \{0, 1\}^n$, where $\{0, 1\}^n$ is the set of all n-bit binary strings. For convenience, u represents vertex u and the string labeling itself. To represent parts of u, we use $p_j(u)$ to denote the j-prefix of u and $s_i(u)$ the suffix of u, i.e., $p_j(u) = u_{n-1}u_{n-2} \cdots u_{n-j}$ and $s_i(u) = u_{i-1}u_{i-2} \cdots u_0$. In the shuffle cubes, the lth 4-bit of u is defined as the consequent four bits starting with the bit u_{4l-2} except the 0th 4-bit is u_1u_0. We use u_4^l to represent the lth 4-bit, i.e., $u_4^0 = u_1u_0$ and $u_4^l = u_{4l+1}u_{4l}u_{4l-1}u_{4l-2}$ for $l \geq 1$. To evaluate the difference between two vertices, we introduce the hamming distance and 4-bit hamming distance. Let u, v be two vertices. The hamming distance between u and v is the number of bits that differ in u and v, denoted by $h(u, v)$. The 4-bit hamming distance between u and v is the number of 4-bits that differ in u and v, denoted by $h_4(u, v)$. To specific the definition, $h(u, v) = |\{i \mid u_i \neq v_i \text{ for } 0 \leq i \leq n-1\}|$ and $h_4(u, v) = |\{j \mid u_4^j \neq v_4^j \text{ for } 0 \leq j \leq \frac{n-2}{4}\}|$. For convenience, we use $h_4^*(u, v)$ to denote $|\{j \mid u_4^j \neq v_4^j \text{ for } 1 \leq j \leq \frac{n-2}{4}\}|$ which does not consider the 0th 4-bit.

To construct the shuffle cube, we use the following four sets: $V_{00} = \{0001, 0010, 0011, 1111\}$, $V_{01} = \{0100, 0101, 0110, 0111\}$, $V_{10} = \{1000, 1001, 1010, 1011\}$, and $V_{11} = \{1100, 1101, 1110, 1111\}$; and use V_{ab}^i to denote the ith member of set V_{ab} for $a, b \in \{0, 1\}$ and $1 \leq i \leq 4$, for example, $V_{10}^2 = 1001$. For ease of exposition, we limit our discussion to $n = 4k + 2$ for $k \geq 0$.

Definition 1 *The n-dimensional shuffle cubes, SQ_n, consists 2^n vertices labeled with n-bit binary strings. For any two vertices u and v, they are adjacent if and only if:*

1. *$u_1u_0 \oplus v_1v_0 \in \{01, 10\}$ and $p_{n-2}(u) = p_{n-2}(v)$; or*
2. *$u_4^j \oplus v_4^j \in V_{u_1u_0}$ for some $j \geq 1$ and $u_4^i = v_4^i$ for any $0 \leq i \neq j \leq k$.* □

For example, the ten neighbors of 0011100100 in SQ_{10} are given by $\underline{0010}100100$, $\underline{0001}100100$, $\underline{0000}100100$, $\underline{1100}100100$, $0011\underline{1000}00$, $0011\underline{1011}00$, $0011\underline{1010}00$, $0011\underline{0110}00$, $00111001\underline{10}$, $00111001\underline{01}$. We can easily find that any two vertices u and v in SQ_n are adjacent only if $h_4(u, v) = 1$. However, the converse is not necessarily true. For example, $u = 0000000000$ and $v = 0000000011$ are not adjacent though $h_4(u, v) = 1$. Thus, the distance between u and v is greater or equal to $h_4(u, v)$.

For ease of construction, we introduce a recursive approach to construct SQ_n in the following. Given $i_1i_2i_3i_4 \in \{0, 1\}^4$, let $G = (V, E)$ be an induced subgraph of SQ_n, where $V = \{u \mid p_4(u) = i_1i_2i_3i_4\}$. Let $\pi(u) = s_{n-4}(u)$ for $u \in V$. Then it is easy to check that π is an isomorphism between G and SQ_{n-4}. So G is isomorphic to SQ_{n-4}. We call such G as an $(n-4)$-*dimensional subcube* of SQ_n and denote it by $SQ_n^{i_1i_2i_3i_4}$. If there is no ambiguity, we simply use *subcubes* as an abbreviation of $(n-4)$-dimensional subcubes. It is apparent that there are 16 disjoint subcubes in SQ_n for $n \geq 6$. Thus, we can recursively construct SQ_n for $n = 4k + 2$ as follows:

1. $SQ_2 = Q_2$.
2. For $n \geq 6$, SQ_n consists of 16 subcubes SQ_{n-4}. For any two vertices u and v in different subcubes, u and v are adjacent if and only if $s_{n-4}(u) = s_{n-4}(v)$ and $p_4(u) \oplus p_4(v) \in V_{u_1u_0}$.

We illustrate SQ_6 in Figure 1. Since there are so many edges in SQ_6 that showing all of them do not make the fashion of connection clear, we only show the edges incident to vertices in SQ_6^{0000} and omit others.

We use programs to verify the following result to simplify our proof since the proof for SQ_6 is tedious.

Lemma 1 *SQ_6 is 4-hamiltonian and 3-hamiltonian connected.*

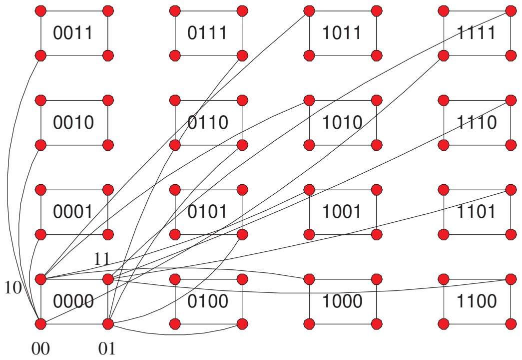

Figure 1: SQ_6.

Thus, the following discussion is for $n \geq 10$.

In our proof, we use fault-hamiltonicity and fault-hamiltonian connectivity as co-operating tools. Moreover, we use the property fault-hamiltonian connectivity to construct major part of hamiltonian cycles or paths in our proof. Let us present out construction scheme first.

For convenience of discussion, we introduce the following terms and notations. Given a vertex $u \in SQ_n^s$ for $s \in \{0,1\}^4$, we use $N(u)$ to denote the set of the neighbors of u. We call the neighbors of u which are not in SQ_n^s the *outer neighbors* of u and use $N^*(u)$ to denote the set of these outer neighbors. Obviously, $|N(u)| = n$ and $|N^*(u)| = 4$. In $SQ_n - F$, we use $N_F(u)$ to denote the neighbors of u and $N_F^*(u)$ to denote the outer neighbors of u, respectively. To represent a path P, we may directly list all the vertices in P such as $\langle v_0, v_1, v_2, \cdots, v_k\rangle$; point out the two endpoints of P such as $\langle v_0, P, v_k\rangle$; or list subpaths of P such as $\langle v_0, P_1, v_t, v_{t+1}, P_2, v_k\rangle$, where P_1 is a path $\langle v_0, v_1, \cdots, v_t\rangle$ and P_2 is a path $\langle v_{t+1}, v_{t+2}, \cdots, v_k\rangle$ for $0 \leq t < k$.

We say that two disjoint subcubes SQ_n^s and SQ_n^t are *adjacent* if there is an edge (u, v) such that $u \in V(SQ_n^s)$ and $v \in V(SQ_n^t)$. We call the edge (u, v) as a bridge between SQ_n^s and SQ_n^t. Furthermore, we say that SQ_n^s and SQ_n^t are *strongly adjacent* if there are at least three bridges between them. Notice that, by definition of the shuffle cubes, any two bridges between the two given subcubes connect to distinct four vertices. We use $E^{s,t}$ to denote the sets of bridges between the two subcubes in SQ_n and use $E_F^{s,t}$ to denote the sets of bridges between the two subcubes in $SQ_n - F$, respectively. Note that $(u, v) \in E_F^{s,t}$ if and only if $\{u, v, (u, v)\} \cap F = \emptyset$. Since $(sx, tx) \in E^{s,t}$ if $s \oplus t \in V_{x_1x_0}$ for $x \in \{0,1\}^{n-4}$, $|E^{s,t}| = 2^{n-4}/4$ or $2 \times 2^{n-4}/4$. So $|E^{s,t}| \geq (n-2) + 3$ for $n \geq 10$, and we have the following property:

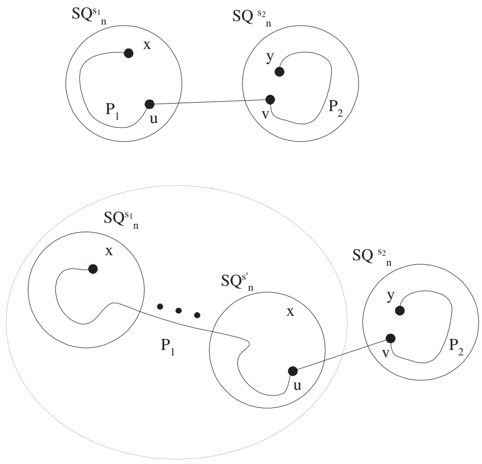

Figure 2: Lemma 2: (a) $|S| = 2$. (b)$|S| \geq 3$.

Proposition 1 *Let $F \subseteq V(SQ_n) \cup E(SQ_n)$ with $|F| \leq n-2$ for $n \geq 10$. Then for any $s \neq t \in \{0,1\}^4$, SQ_n^s and SQ_n^t are still strongly adjacent in $SQ_n - F$, i.e., $|E_F^{s,t}| \geq 3$.*

Lemma 2 *Suppose that $n \geq 10$, $S \subseteq \{0,1\}^4$ with $|S| \geq 2$, and $F \subseteq V(SQ_n) \cup E(SQ_n)$ with $|F| \leq n-2$. And assume that $SQ_n^s - F$ is still hamiltonian connected for each $s \in S$. Then for any $x \in V(SQ_n^{s_1})$ and $y \in V(SQ_n^{s_2})$ with $s_1 \neq s_2 \in S$, there is a hamiltonian path of $SQ_n^S - F$ between x and y.*

Proof. We prove this lemma by induction on $|S|$.

(1) $|S| = 2$**.** Then $S = \{s_1, s_2\}$. By Proposition 1, $|E_F^{s_1,s_2}| \geq 3$. So there is an edge $(u, v) \in E_F^{s_1,s_2}$ such that $u \neq x \in V(SQ_n^{s_1})$ and $v \neq y \in V(SQ_n^{s_2})$. By assumption that each $SQ_n^s - F$ is hamiltonian connected, there are a hamiltonian path P_1 of $SQ_n^{s_1} - F$ from x to u and a hamiltonian path P_2 of $SQ_n^{s_2} - F$ from v to y. Then $\langle x, P_1, u, v, P_2, y\rangle$ forms a hamiltonian path of $SQ_n^S - F$ between x and y.

(2) $|S| \geq 3$**.** Assume that the lemma is true for all S' with $2 \leq |S'| < |S|$. Since $|S| \geq 3$, there is an $s' \in S$ with $s' \neq s_1, s_2$. By Proposition 1,

$|E_F^{s',s_2}| \geq 3$. There is an edge $(u,v) \in E_F^{s',s_2}$ such that $u \in V(SQ_n^{s'})$ and $v \neq y \in V(SQ_n^{s_2})$. Then, by induction hypothesis, there is a hamiltonian path P_1 of $SQ_n^{S-\{s_2\}} - F$ from x to u. And by assumption there is a hamiltonian path P_2 of $SQ_n^{s_2} - F$ from v to y. Then $\langle x, P_1, u, v, P_2, y\rangle$ forms a hamiltonian path of $SQ_n^S - F$ between x and y. $\square$

Note that in Lemma 2, the hamiltonian path of SQ_n^S does not cross any edge in $E_F^{s_1,s_2}$ if $|S| \geq 3$. The following lemma can be obtained directly from the definition.

Lemma 3 *Let G be a hamiltonian graph and f be a vertex or an edge in G. Then there is a hamiltonian path in $G - f$.*

Consider the location of f. If f is not on some hamiltonian cycle of G, G contains a hamiltonian cycle and obviously, contains a hamiltonian path. If f is on all hamiltonian cycles of G, then removing f from these cycles results in hamiltonian paths of G. The lemma follows. By this lemma, if G is k-hamiltonian, $G - F$ contains a hamiltonian path for $F \subseteq V(G) \cup E(G)$ and $|F| \leq k+1$.

The following lemma concerns the fault distribution in the shuffle cube. It save us from dealing with the most difficult cases of the main result. For ease of discussion, we first define some terms and notations here. Let $n = 4k+2$ for some $k \geq 1$. For $1 \leq i \leq k$ and $s \in \{0,1\}^4$, we define the $(n-4)$-dimensional subcube $SQ_n^{i:s}$ as the induced subgraph of $V(SQ_n^{i:s}) = \{v \in V(SQ_n) \mid v_4^i = s\}$. Obviously, $SQ_n^{k:s}$ is SQ_n^s. Extending this notation, we define $SQ_n^{i_1,i_2,\cdots,i_l:s_1,s_2\cdots,s_l}$ as the induced subgraph of $V(SQ_n^{i_1,i_2,\cdots,i_l:s_1,s_2\cdots,s_l}) = \{v \in V(SQ_n) \mid v_4^{i_1} = s_1, v_4^{i_2} = s_2, \cdots, v_4^{i_l} = s_l\}$. Let F be any faulty set of SQ_n. We use $F(SQ_n^{i_1,i_2,\cdots,i_l:s_1,s_2\cdots,s_l})$ to denote the set of faults in $SQ_n^{i_1,i_2,\cdots,i_l:s_1,s_2\cdots,s_l}$.

Lemma 4 *(Fault distribution)*
Given $n = 4k+2$ for $k \geq 2$. Let $F \subseteq V(SQ_n) \cup E(SQ_n)$ with $|F| \geq n-3$. There is another faulty set F' such that $SQ_n - F \cong SQ_n - F'$ and $|F'(SQ_n^s)| \leq |F| - 2$ for all $s \in \{0,1\}^4$.

Proof. Let $E' = \{(u,v) \in F \mid u \text{ or } v \in F\}$ and $F_0 = F - E'$. Obviously, $SQ_n - F_0 \cong SQ_n - F$ since the induced subgraph of $SQ_n - F_0$ results in removing all edges in E' from SQ_n.

First, we show that there exists an index j with $1 \leq j \leq k$ such that $|F_0(SQ_n^{j:s})| \leq |F| - 2$ for all $s \in \{0,1\}^4$. Suppose on the contrary that for each index $1 \leq i \leq k$, there exists $s_i \in \{0,1\}^4$ such that $|F_0(SQ_n^{i:s_i})| \geq |F| - 1$. We shall prove that $|F_0(SQ_n^{1,2,\cdots,t:s_1,s_2,\cdots,s_t})| \geq |F| - t$ for any t with $1 \leq t \leq k$ by induction: Since $|F_0(SQ_n^{1:s_1})| \geq |F| - 1$, the statement holds for $t = 1$. Assume that the statement holds for any t' with $1 \leq t' < t \leq k$. By inclusion-exclusion principle,

$$\begin{aligned}
&|F_0(SQ_n^{1,2,\cdots,t:s_1,s_2,\cdots,s_t})| \\
= \;& |F_0(SQ_n^{1,2,\cdots,t-1:s_1,s_2,\cdots,s_{t-1}})| + |F_0(SQ_n^{t:s_t})| \\
& -|F_0(SQ_n^{1,2,\cdots,t-1:s_1,s_2,\cdots,s_{t-1}}) \cup F_0(SQ_n^{t:s_t})| \\
\geq \;& |F| - (t-1) + |F| - 1 - |F| \\
= \;& |F| - t
\end{aligned}$$

Thus, $|F_0(SQ_n^{1,2,\cdots,k:s_1,s_2,\cdots,s_k})| \geq |F| - k \geq (4k+2-3) - k = 3k - 1 \geq 5$ for $k \geq 2$. However, $SQ_n^{1,2,\cdots,k:s_1,s_2,\cdots,s_k} \cong Q_2$, so there are at most 4 faults in F_0. We get a contradiction. Thus, there is an index j with $1 \leq j \leq k$ such that $|F_0(SQ_n^{j:s})| \leq |F| - 2$ for all $s \in \{0,1\}^4$.

Next, we construct F'. For $1 \leq i \leq k$, let π_i be a mapping of $V(SQ_n)$ into itself such that $\pi_i(u) = u_{4i+1}u_{4i}u_{4i-1}u_{4i-2}$ $u_{n-5}u_{u-6}\cdots u_{4i+2}u_{n-1}u_{n-2}u_{n-3}u_{n-4}u_{4i-3}\cdots u_1u_0$, i.e., exchanging the positions of u_4^i and u_4^k. Clearly, π_i is an automorphism of SQ_n. Let $F' = V' \cup E'$ be the subset of $V \cup E$ such that $V' = \{\pi_j(v) \mid v \in F_0\}$ and $E' = \{(\pi_j(u), \pi_j(v)) \mid (u,v) \in F_0\}$. Then F' is the faulty set satisfying the given condition. $\square$

By this lemma, we may consider those faulty sets F satisfying that $|F(SQ_n^s)| \leq |F| - 2$ for all $s \in \{0,1\}^4$ in our proof of main result.

3 Main result

In this section, we discuss the fault hamiltonicity of the shuffle cubes which is the main result of this paper. In the two lemmas, we shall discuss all faulty set $F \subseteq V(SQ_n) \cup E(SQ_n)$ with $|F| \leq n-2$ or $n-3$, respectively. However, it is sufficient to prove the two lemmas for the cases that $|F| = n-2$ or $n-3$, respectively. For example, for the first lemma, if $|F| < n-2$, we may arbitrarily choose $n-2-|F|$ edges as extra faulty edges to make a faulty set F' satisfying $|F'| = n-2$. Then if $SQ_n - F'$ contains a hamiltonian cycle, $SQ_n - F$ certainly contains one, too.

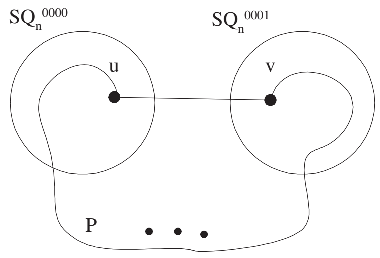

Figure 3: Lemma 5: Case 1.

Lemma 5 *Suppose that SQ_{n-4} is $(n-6)$-hamiltonian and $(n-7)$-hamiltonian connected for some $n=4k+2$ and $k \geq 2$. Then SQ_n is $(n-2)$-hamiltonian.*

Proof. Let $F \subseteq V(SQ_n) \cup E(SQ_n)$ be any faulty set with $|F| = n-2$. For $s \in \{0,1\}^4$, we use $F(SQ_n^s)$ to denote the set of faults in SQ_n^s, i.e., $F(SQ_n^s) = F \cap (V(SQ_n^s) \cup E(SQ_n^s))$. By Lemma 4, we may assume that $|F(SQ_n^s)| \leq n-4$ for all $s \in \{0,1\}$. Consider the following cases:

Case 1. $|F(SQ_n^s)| \leq n-7$ for each $s \in \{0,1\}^4$. By induction hypothesis, each subcube is still hamiltonian connected. By Proposition 1, there is an edge $(u,v) \in E_F^{0000,0001}$ such that $u \in V(SQ_n^{0000})$ and $v \in V(SQ_n^{0001})$. By Lemma 2, there is a hamiltonian path P of $SQ_n - F$ between u and v. Thus, $\langle u, P, v, u \rangle$ is a hamiltonian cycle of $SQ_n - F$. And so $SQ_n - F$ is still hamiltonian.

Case 2. $|F(SQ_n^s)| = n-6$ for some $s \in \{0,1\}^4$. Then there are 4 faults outside SQ_n^s. Since each subcube is $(n-6)$-hamiltonian and $(n-7)$-hamiltonian connected and $n \geq 10$, there is at most one subcube not being hamiltonian connected in $SQ_n - F$ when $n = 10$, and all subcubes are hamiltonian connected when $n > 10$.

Subcase 2.1. $|F(SQ_n^t)| \leq n-7$ for each $t \neq s$. Thus, all the other subcubes are still hamiltonian connected. Let C_s be a hamiltonian cycle of $SQ_n^s - F$. Since the number of vertices on C_s is at least $2^{n-4} - n + 6 > 4$, there exit two vertices u and v adjacent on C_s such that u and v are still connected to two different outer neighbors in $SQ_n - F$. Thus, there are two edges $(u,u') \in E_F^{s,s_1}$ and $(v,v') \in E_F^{s,s_2}$ such that $s_1 \neq s_2$. Let $S = \{0,1\} - \{s\}$. By Lemma 2, there is a hamiltonian path P of $SQ_n^S - F$ between u' and v'. Let $C_s = \langle u, P_s, v, u \rangle$. Then $\langle u, P_s, v, v', P, u', u \rangle$ is a hamiltonian cycle of $SQ_n - F$.

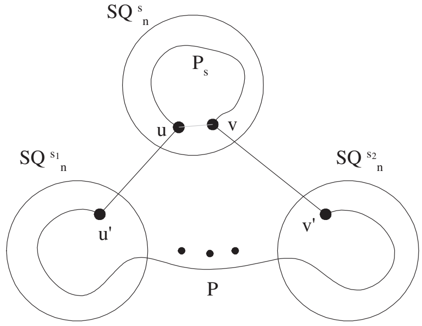

Figure 4: Lemma 5: Subcase 2.1.

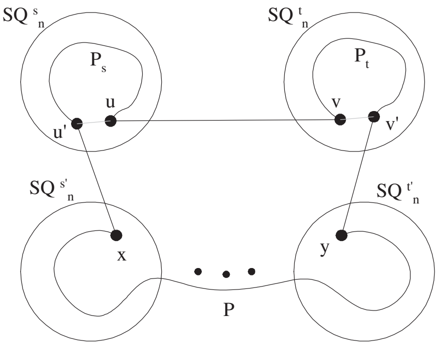

Figure 5: Lemma 5: Subcase 2.2.

Subcase 2.2. There exists a subcube $|F(SQ_n^t)| > n-7$ for $t \neq s$. Then $n = 10$, $|F(SQ_n^t)| = 4$, and there is no fault outside SQ_n^s and SQ_n^t. By Proposition 1, there is an edge $(u,v) \in E_F^{s,t}$, where $u \in V(SQ_n^s)$ and $v \in V(SQ_n^t)$. Since $SQ_n^s - F$ and $SQ_n^t - F$ are hamiltonian, let C_s and C_t be two hamiltonian cycles of $SQ_n^s - F$ and $SQ_n^t - F$, respectively. Certainly, u and v are on C_s and C_t, respectively, so we may let $C_s = \langle u, P_s, u', u \rangle$ and $C_t = \langle v, v', P_t, v \rangle$. Since there are four outer neighbors of each of u' and v', there are two edges $(u',x) \in E_F^{s,s'}$ and $(v',y) \in E_F^{t,t'}$ such that s, s', t, t' are distinct. Let $S = \{0,1\}^4 - \{s,t\}$. By Lemma 2, there is a hamiltonian path P of $SQ_n^S - F$ between x and y. Then $\langle u, P_s, u', x, P, y, v', P_t, v, u \rangle$ is a hamiltonian cycle of $SQ_n - F$.

Case 3. $|F(SQ_n^s)| = n-5$ for some $s \in \{0,1\}^4$. Then there are 3 faults outside SQ_n^s. So all the other subcubes are hamiltonian and hamiltonian connected. By Lemma 3, there is a hamiltonian path of $SQ_n^s - F$. Let $\langle u, P_s, v \rangle$ be such the path. Since there are 3 faults outside SQ_n^s and each of u and v has four outer neighbors, there exit two

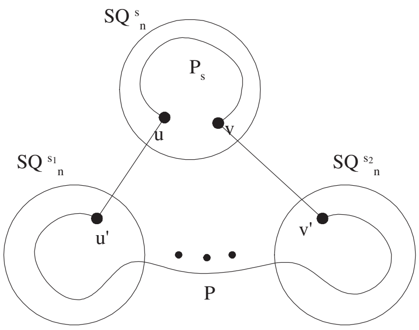

Figure 6: Lemma 5: Case 3.

edges $(u, u') \in E_F^{s,s_1}$ and $(v, v') \in E_F^{s,s_2}$ such that $s_1 \neq s_2$. Then the consequent construction for the hamiltonian cycle of $SQ_n - F$ is similar to Subcase 2.1.

Case 4. $|F(SQ_n^s)| = n-4$ for some $s \in \{0,1\}^4$. Then there are 2 faults outside SQ_n^s. So all the other subcubes are hamiltonian and hamiltonian connected. Let f be a fault in $F(SQ_n^s)$ and $F_1 = F - \{f\}$. Then $|F_1(SQ_n^s)| = n-5$ and by Lemma 3, there is a hamiltonian path of $SQ_n^s - F_1$. Let $\langle u, P_s, v\rangle$ be the path. Now consider $F_1 \cup \{f\}$, which is exactly F. If the path P_s is still a path in $SQ_n - F$, i.e., f is u, v, or an edge which is not on P_s, then the proof is similar to Case 3.

Otherwise, P_s is divided into two disjoint paths in $SQ_n - F$. Suppose the two disjoint paths are $\langle u, P_1, u'\rangle$ and $\langle v', P_2, v\rangle$. Note that P_1 or P_2 may be of length 0, i.e., containing only the vertex u or the vertex v. Since each of u, u', v, and v' has four outer neighbors and there are 2 faults outside SQ_n^s, by careful selection, we can find four edges $(u, x) \in E_F^{s,s_1}$, $(u', x') \in E_F^{s,s_2}$, $(v, y) \in E_F^{s,s_3}$, and $(v', y') \in E_F^{s,s_4}$ such that s_1, s_2, s_3, and s_4 are distinct. Let $S_1 = \{s_1, s_3\}$ and $S_2 = \{0,1\}^4 - \{s, s_1, s_3\}$. By Lemma 2, there are a hamiltonian path P_3 of $SQ_n^{S_1} - F$ between x and y and a hamiltonian path P_4 of $SQ_n^{S_2} - F$ between x' and y'. Then $\langle u, P_1, u', x', P_4, y', v', P_2, v, y, P_3, x, u\rangle$ forms a hamiltonian cycle of $SQ_n - F$.

Hence, the lemma follows.

□

Lemma 6 *Suppose that SQ_{n-4} is $(n-6)$-hamiltonian and $(n-7)$-hamiltonian connected for $n = 4k+2$ and $k \geq 2$. Then SQ_n is $(n-3)$-hamiltonian connected.*

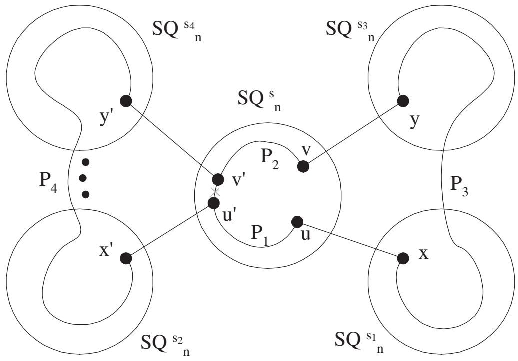

Figure 7: Lemma 5: Case 4.

Proof. Let $F \subseteq V(SQ_n) \cup E(SQ_n)$ be any faulty set with $|F| = n-3$. And let $x \in V(SQ_n^s)$ and $y \in V(SQ_n^t)$ be any two vertices in $SQ_n - F$ for $s, t \in \{0,1\}^4$. We shall prove that there is a hamiltonian path of $SQ_n - F$ between x and y. For $s \in \{0,1\}^4$, we also use $F(SQ_n^r)$ to denote the set of faults in SQ_n^r. By Lemma 4, we may assume that $|F(SQ_n^r)| \leq n-5$ for all $r \in \{0,1\}^4$. Consider the following cases:

Case 1. $|F(SQ_n^r)| \leq n-7$ for each $r \in \{0,1\}^4$. By induction hypothesis, each subcube is still hamiltonian connected. Consider two subcases:

Subcase 1.1. $s = t$, i.e, x and y are in the same subcube. Since each subcube is hamiltonian connected, there is a hamiltonian path P_s of $SQ_n^s - F$ between x and y. Let $f = |F \cap V(SQ_n^s)|$. Then there are $2^{n-4} - f$ vertices on P_s and at most $n-3-f$ faults outside SQ_n^s. Since $2^{n-4} - f > 2(n-3-f)$, there are two vertices u and v adjacent on P_s which connect four outer neighbors in $SQ_n - F$. So there exit two edges $(u, u') \in E_F^{s,s_1}$ and $(v, v') \in E_F^{s,s_2}$ such that $s_1 \neq s_2$. Let $S = \{0,1\}^4$. By Lemma 2, there is a hamiltonian path P of $SQ_n^S - F$ between u' and v'. Suppose that $P_s = \langle x, P_1, u, v, P_2, y\rangle$. Note that x, u, v, y are not necessarily distinct. Then $\langle x, P_1, u, u', P, v', v, P_2, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

Subcase 1.2. $s \neq t$, i.e, x and y are in different subcubes. By Lemma 2, there is a hamiltonian path of $SQ_n - F$ between x and y.

Case 2. $|F(SQ_n^r)| = n-6$ for some $r \in \{0,1\}^4$. Then $SQ_n^r - F$ is hamiltonian. Let C_r be a hamiltonian cycle of $SQ_n^r - F$. On the other hand, there are 3 faults outside SQ_n^r. Thus, all the other sub-

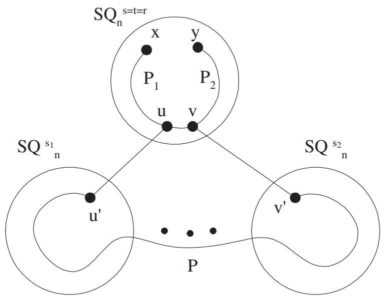

Figure 8: Lemma 6: Subcase 1.1.

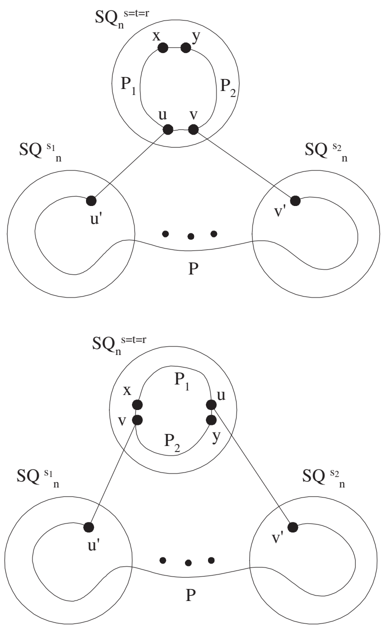

Figure 9: Lemma 6: Subcase 2.1.

cubes are still hamiltonian and hamiltonian connected in $SQ_n - F$. Consider the following four cases:

Subcase 2.1. $s = t = r$. Suppose that x and y are adjacent on C_r, i.e., $C_s = \langle x, P_s, y, x\rangle$ for some path P_r in $SQ^r_n - F$. Then the proof is similar to subcase 1.1. Suppose that x and y are not adjacent on C_r, i.e., $C_s = \langle x, P_1, u, y, P_2, v, x\rangle$, where x, u, y, v are distinct. Since each of u and v has four outer neighbors and there are 3 faults outside SQ^r_n, there exit two edges $(u, u') \in E^{r,s_1}_F$ and $(v, v') \in E^{r,s_2}_F$ such that $s_1 \neq s_2$. Let $S = \{0, 1\}^4 - \{r\}$. By Lemma 2, there is a hamiltonian path P of $SQ^S_n - F$ between u' and v'. Then $\langle x, P_1, u, u', P, v', v, P_2, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

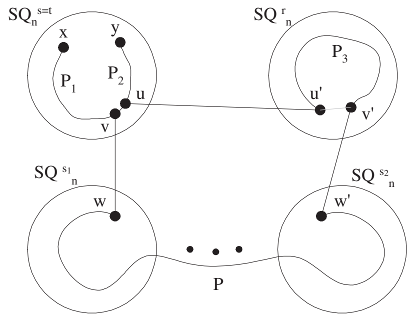

Figure 10: Lemma 6: Subcase 2.2.

Subcase 2.2. $s = t \neq r$. Then SQ^s_n is hamiltonian connected and so there is a hamiltonian path P_s of SQ^s_n between x and y. By Proposition 1, there is an edge $(u, u') \in E^{s,r}_F$ such that u is on P_s, u' is on C_r, and $u \neq x, y$. Clearly, there are two neighbors of u on P_s and two neighbors of u' on C_r, and each of them has four neighbors. Since there are 3 faults outside SQ^r_n, there exit a neighbor v of u on P_s and a neighbor v' of u' on C_r such that each of v and v' has three outer neighbors in $SQ_n - F$. Thus, there are two edges $(v, w) \in E^{s,s_1}_F$ and $(v', w') \in E^{r,s_2}_F$ such that s, r, s_1, s_2 are distinct. Let $S = \{0, 1\} - \{s, r\}$. By Lemma 2, there is a hamiltonian path P of $SQ^S_n - F$ between w and w'. Assume that $P_s = \langle x, P_1, u, v, P_2, y\rangle$ and $C_r = \langle u', P_3, v', u'\rangle$. Then $\langle x, P_1, u, u', P_3, v', w', P, w, v, P_2, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

Subcase 2.3. $s = r \neq t$. (The case that $t = r \neq s$ is similar.) Since there are two neighbors of x on C_r and there are 3 faults outside SQ^r_n, there exits a neighbor u of x on C_r such that u has two outer neighbors in $SQ_n - F$. Then there is an edge $(u, v) \in E^{r,s_1}_F$ such that $s_1 \neq t$. Let $S = \{0, 1\} - \{r\}$. By Lemma 2, there is a hamiltonian path P of SQ^S_n between v and y. Assume that $C_r = \langle x, P_1, u, x\rangle$. Then $\langle x, P_1, u, v, P, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

Subcase 2.4. r, s, and t are distinct. By Proposition 1, there is en edge $(u, u') \in E_F(s, r)$ such that $u \neq x \in V(SQ^s_n)$ and $u' \in V(SQ^r_n)$. Since SQ^s_n is hamiltonian connected, there is a hamiltonian path P_s of $SQ^s_n - F$ between x and u. On the other hand, there are two neighbors of u' on C_r and there are 3 faults outside SQ^r_n. So there is a neighbor v' of u' on C_r such that v' has three outer neighbors. Then there exits an edge $(v', v) \in E^{r,s'}_F$ such that s, r, t, and s' are dis-

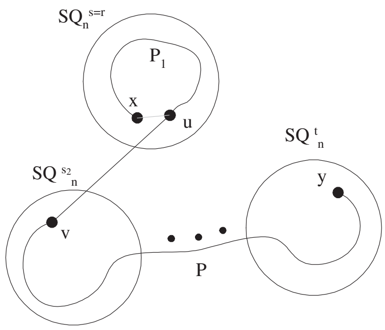

Figure 11: Lemma 6: Subcase 2.3.

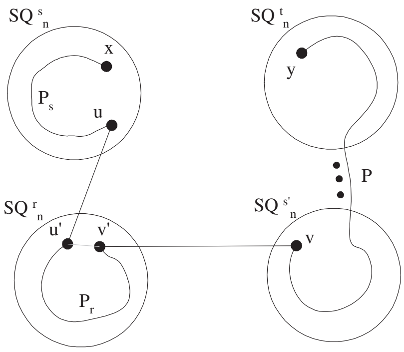

Figure 12: Lemma 6: Subcase 2.4.

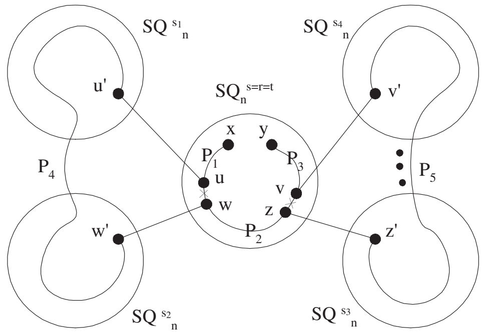

Figure 13: Lemma 6: Subcase 3.1.

tinct. Let $S = \{0,1\}^4 - \{s, r\}$. By Lemma 2, there is a hamiltonian path P of $SQ_n^S - F$ between v and y. Assume that $C_r = \langle u', P_r, v', u' \rangle$. Then $\langle x, P_s, u, u', P_r, v', v, P, y \rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

Case 3. $|F(SQ_n^r)| = n-5$ for some $r \in \{0,1\}^4$. By Lemma 3, there is a hamiltonian path P_r of $SQ_n^r - F$ between two vertices, say u and v. Note that u and v are determined by the given faulty set F. There are 2 faults outside SQ_n^r. Thus, the other subcubes are still hamiltonian and hamiltonian connected in $SQ_n - F$. Consider the following four cases:

Subcase 3.1. $s = t = r$. Suppose that $f_1, f_2 \in F(SQ_n^r)$. Then by induction hypothesis, in $SQ_n^r - (F - \{f_1, f_2\})$, there is a hamiltonian path P_r between x and y. Clearly, f_1 and f_2 may be on P_r or not. Here we only discuss the case that f_1 and f_2 are both on P_r since the other cases can be handled by the same method.

Suppose that f_1 and f_2 are on P_r. Then P_r is divided into three parts: $\langle x, P_1, u \rangle$, $\langle w, P_2, z \rangle$, and $\langle v, P_3, y \rangle$. Now consider the order of $N_F^*(u)$, $N_F^*(v)$, $N_F^*(w)$, and $N_F^*(z)$. Without loss of generality, assume that $N_F^*(u) \leq N_F^*(v) \leq N_F^*(w) \leq N_F^*(z)$. Since there are only two faults outside SQ_n^r, by the assumption, there exit four edges $(u, u') \in E_F^{r,s_1}$, $(w, w') \in E_F^{r,s_2}$, $(z, z') \in E_F^{r,s_3}$, and $(v, v') \in E_F^{r,s_4}$ such that s_1, s_2, s_3, and s_4 are distinct. Let $S = \{s_1, s_2\}$ and $S' = \{0,1\}^4 - \{r, s_1, s_2\}$. By Lemma 2, there are a hamiltonian path P_4 of $SQ_n^S - F$ between u' and w' and a hamiltonian path P_5 of $SQ_n^{S'} - F$ between z' and v'. Then $\langle x, P_1, u, u', P_4, w', w, P_2, z, z', P_5, v', v, P_3, y \rangle$ forms a hamiltonian path of $SQ_n - F$.

Subcase 3.2. $s = t \neq r$. Then $SQ_n^s - F$ is hamiltonian connected and there is a hamiltonian path P_s between x and y. Now consider the hamiltonian path P_r in $SQ_n^r - F$. Without loss of generality, assume that $2 \leq |N_F^*(u)| \leq |N_F^*(v)|$. Then there is an edge $(u, u') \in E_F^{r,s_1}$ such that $s_1 \neq s$. Since $|E^{s,s_1}| \geq 2^{n-4} \geq 6$ for $n \geq 10$ and there are 2 faults outside SQ_n^r, $|E_F^{s,s_1}| \geq 4$ and there is an edge $(w, w') \in E_F^{s,s_1}$ such that $w \in V(SQ_n^s - F) - \{x, y\}$ and $w' \in V(SQ_n^{s_1} - F) - \{u'\}$. Then there is a hamiltonian path P_{s_1} of $SQ_n^{s_1} - F$ between u' and w'. Let $P_s = \langle x, P_1, a, w, z, P_2, y \rangle$, where a may be x or z may be y. Suppose that $N_F^*(v) = 3$. Then $N_F^*(u) = 3$, so $N_F^*(z) = 4$. There are two edges $(v, v') \in E_F^{r,s_2}$ and $(z, z') \in E_F^{r,s_3}$ such that s, s_1, s_2, and s_3 are distinct. Otherwise, $N_F^*(v) = 4$. Assume that $|N_F^*(z)| \geq |N_F^*(a)|$. Since there are only 2 faults outside SQ_n^r, $N_F^*(z) \geq 3$. Then there still exit two edges $(z, z') \in E_F^{r,x_3}$ and $(v, v') \in E_F^{r,s_2}$ such that s, s_1, s_2, and s_3 are distinct. Let $S = \{0,1\}^4 - \{r, s, s_1\}$. By Lemma 2, there is a hamiltonian path P_3 of $SQ_n^S - F$ between v' and z'. Thus, $\langle x, P_1, a, w, w', P_{s_1}, u', u, P_r, v, v', P_3, z', z, P_2, y \rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

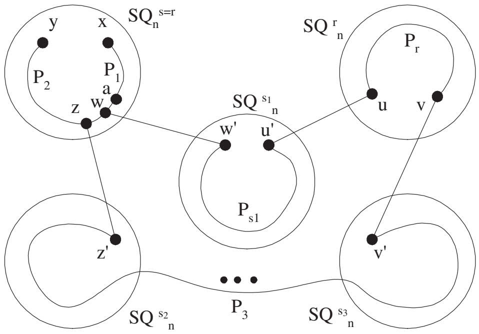

Figure 14: Lemma 6: Subcase 3.2.

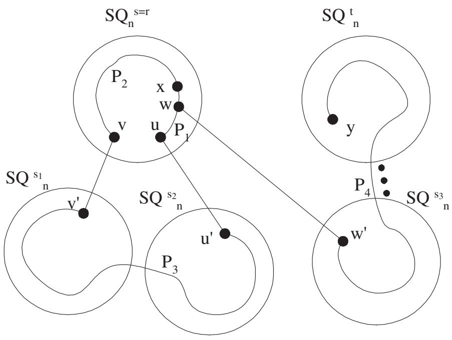

Figure 15: Lemma 6: Subcase 3.3.

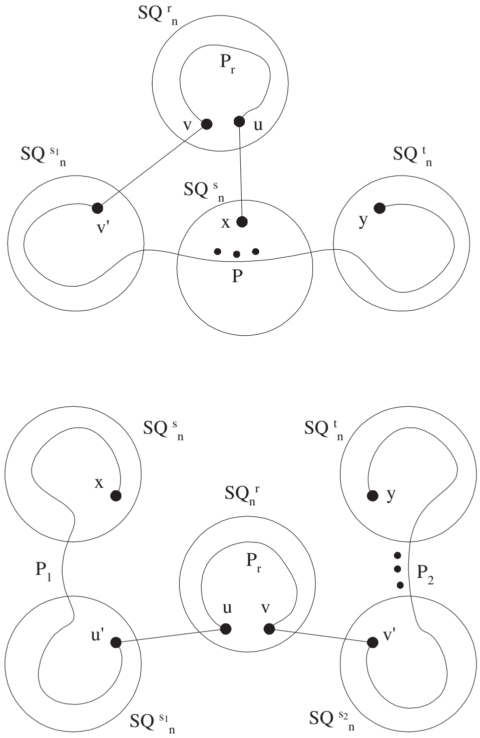

Figure 16: Lemma 6: Subcase 3.4.

Subcase 3.3. $s = r \neq t$. (The case that $t = r \neq s$ is similar.) Suppose that x is u (or v). Since there are only 2 faults outside SQ_n^r, there is an edge (v, v') (or (u, v')) $\in E_F^{r,s'}$ such that $s' \neq t$. Then it is similar to Subcase 2.3 to construct a hamiltonian path of $SQ_n - F$ between x and y.

Suppose that x is neither u nor v. Assume $P_r = \langle u, P_1, w, x, P_2, v\rangle$, where w may be u. Without loss of generality, assume that $2 \leq |N_F^*(u)| \leq |N_F^*(v)| \leq |N_F^*(w)|$. Then $|N_F^*(v)| \geq 3$ and $|N_F^*(w)| \geq 4$. Thus, there are three edges $(u, u') \in E_F^{r,s_1}$, $(v, v') \in E_F^{r,s_2}$, and $(w, w') \in E_F^{r,s_3}$ such that s_1, s_2, s_3, and t are distinct. Note that we select these edges according to the order of $|N_F^*(u)|$, $|N_F^*(v)|$, and $|N_F^*(w)|$. Let $S = \{s_1, s_2\}$ and $S' = \{0,1\}^4 - \{r, s_1, s_2\}$. By Lemma 2, there are a hamiltonian path P_3 of $SQ_n^S - F$ between u' and v' and a hamiltonian path P_4 of $SQ_n^{S'} - F$ between w' and y. Then $\langle x, P_2, v, v', P_3, u', u, P_1, w, w', P_4, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

Subcase 3.4. r, s, and t are distinct. Without loss of generality, assume that $2 \leq |N_F^*(u)| \leq |N_F^*(v)|$. Suppose that x (or y) $\in N_F^*(u)$. (We concentrate on the case $x \in N_F^*(u)$.) Let $F_1 = (F - F(SQ_n^r)) \cup \{x\}$. Then $|F_1| = 3 \leq n-2$. Since $|N_F^*(v)| \geq 3$, there is an edge $(v, v') \in E_F^{r,s_1}$ such that $s_1 \neq s, t$. Let $S = \{0,1\}^4 - \{r\}$. By Lemma 2, there is a hamiltonian path P of $SQ_n^S - F_1$ between v' and y. Then $\langle x, u, P_r, v, v', P, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y.

Suppose that $x, y \notin N_F^*(u)$ and u is adjacent to SQ_n^s or SQ_n^t. Assume that u is still adjacent to SQ_n^s in $SQ_n - F$. Then there is an edge $(u, u') \in E_F^{r,s}$ such that $u' \neq x$. And the consequent proof is similar to Subcase 2.4.

Suppose that u is neither adjacent to SQ_n^s nor adjacent to SQ_n^t. Since $|N_F^*(v)| \geq 3$, there is an edge $(v, v') \in E_F^{r,s_1}$ such that $s_1 \neq s, t$. And then there is an edge $(u, u') \in E_F^{r,s_2}$ such that $s_2 \neq s, t, s_1$. Let $S = \{s, s_2\}$ and $S' = \{0,1\}^4 - \{r, s, s_2\}$. By Lemma 2, there are a hamiltonian path P_1 of $SQ_n^S - F$ between u' and x and a hamiltonian path P_2 of $SQ_n^{S'} - F$ between v' and y. Then $\langle x, P_1, u', u, P_r, v, v', P_2, y\rangle$ forms a hamiltonian path of $SQ_n - F$ between x and y. $\square$

Using Lemma 1 as the base case and Lemma 5, 6 as the induction step, we have the following theorem.

Theorem 1 *The n-dimensional shuffle cubes SQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for $n \geq 6$.*

References

[1] S. Abraham and K. Padmanabhan, "The twisted cube topology for multiprocessors: A study in network asymmetry," *Journal of Parallel and Distributed Computing,* vol. 13, pp. 104-110, 1991.

[2] J.A. Bondy and U.S.R. Murty, Graph theory with applications, North Holland, New York, 1980.

[3] M.Y. Chan and S.J. Lee, "On the existence of hamiltonian circuits in faulty hypercubes," *SIAM J. Discrete Mathematics,* vol. 4, no. 4, pp. 511–527, Nov. 1991.

[4] P. Cull and S.M. Larson, "The Möbius Cubes," *IEEE Trans. Computers,* vol. 44, no. 5, May 1995.

[5] K.Efe, "A Variation on the Hypercube with lower Diameter," *IEEE Trans. Computers,* vol. 40, no. 11, pp. 1312-1316, Nov. 1991.

[6] K. Efe, "The crossed cube architecture for parallel computing," *IEEE Trans. Parallel Distruibuted Systems,* vol. 3, no. 5, pp. 513 -524, Sept. 1992.

[7] P. A. J. Hilbers, R. J. M. Koopman, and J. L. A. Van de Snepscheut, The twisted cube, *Parallel Archetectures Languages Europe,* Lectue Notes in Computer Science, pp. 152–158, June 1987.

[8] S.Y. Hsieh, G.H. Chen, and C.W. Ho, "Longest fault-free paths in star graphs with edge faults," *IEEE Trans. Computers,* vol. 50, no. 9, pp. 960–971, Sep. 2001.

[9] W.T. Huang, J.M. Tan, C.N. Hung, and L.H. Hsu, "Fault-tolerant hamiltonicity of twisted cubes", *Jounal of Parallel and Distributed Computing,* Accepted, 2001.

[10] S. Latifi, S.Q. Zheng, and N. Bagherzadeh, "Optimal ring embedding in hypercubes with faulty links," *Proceedings of the IEEE Symposium on Fault-Tolerant Computing,* 42, pp. 178–184, 1992.

[11] T.K. Li, J.M. Tan, L.H. Hsu, and T.Y. Sung, "The Shuffle Cubes and Their Generalization," *Information Processing Letters,* vol. 77, pp. 35–41, Jan 2001.

[12] T.K. Li, J.M. Tan, and L.H. Hsu, "The wide diameter of the shuffle cube," *Proceedings of the Nineteenth IASTED International Conference on Applied Informatics,* pp. 402–407, 2001.

[13] R.A. Rowley and B. Bose, "Fault-tolerant ring embedding in de-Bruijn networks," *IEEE Trans. Computers,* vol. 42, no. 12, pp. 1480–1486, Dec. 1993.

[14] Y. Saad and M.H. Schultz, "Topological Properties of Hypercubes," *IEEE Trans. Computers,* vol. 37, no. 7, pp.867-872, July 1988.

[15] Y.C. Tseng, S.H. Chang, and J.P. Sheu, "Fault-tolerant ring embedding in a star graph with both link and node failure," *IEEE Trans. Parallel and Distributed Systems*, 1185-1195, 1997.