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UNIDIRECTIONAL (n, k)-STAR GRAPHS

EDDIE CHENG and MARC J. LIPMAN
Department of Mathematics and Statistics
Oakland University, Rochester, MI 48309, USA

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Arrangement graphs [14] and (n, k)-star graphs[11] were introduced as generalizations of star graphs[1]. They were introduced to provide a wider set of choices for the order of topologically attractive interconnection networks. Unidirectional interconnection networks are more appropriate in many applications. Cheng and Lipman[5], and Day and Tripathi[17] studied the unidirectional star graphs, and Cheng and Lipman[7] studied orientation of arrangement graphs. In this paper, we show that every (n, k)-star graph can be oriented to a maximally arc-connected graph when the regularity of the graph is even. If the regularity is odd, then the resulting directed graph can be augmented to a maximally arc-connected directed graph by adding a directed matching. In either case, the diameter of the resulting directed graph is small. Moreover, there exists a simple and near-optimal routing algorithm.

Keywords: Interconnection networks, (n, k)-star graphs, arc-connectivity, orientation.

1. Introduction

Directed interconnection networks are important. This area has generated many research papers[5,6,10,13,17,20]. In particular, Chou and Du[13] proposed to use the unidirectional hypercubes as the basis for high speed networking. For a more general model, we refer the reader to [10] for an architectural model for the studies of unidirectional graph topologies and a specific application, which also includes a comparison of the diameters among some unidirectional interconnection networks.

The star graph, proposed by Akers, Harel and Krishnamurthy[1], has many advantages over the hypercube, such as lower degree and a smaller diameter. An orientation of the star graph was proposed by Day and Tripathi[17], and they gave an efficient near-optimal distributed routing algorithm for it. One of the main criteria of a good interconnection network topology is that it is maximally edge-connected. So the ideal situation is for a unidirectional graph topology to have the highest possible arc-connectivity. Indeed Jwo and Tuan[20] showed that the unidirectional hypercube proposed by Chou and Du[13] has this important property. Since the star graph was introduced as a competitive alternative to the hypercube, it is necessary that an orientation for the star graph has that same property for it to remain competitive with the hypercube. Indeed, Cheng and Lipman[5] showed that the orientation for the star graphs given by Day and Tripathi[17] has this property.

The main drawback of the star graphs is related to its number of vertices: $n!$ for an n-dimensional star graph. For example, the smallest star graph with at least 6000 vertices is a graph with 40320 vertices. Two classes of "nice" graphs were proposed to solve this problem. The first class is the arrangement graphs introduced by Day and Tripathi[14], and the second class is the (n,k)-star graphs introduced in Chiang and Chen[11]. Both are families of undirected graphs that include the star graphs. The "older" arrangement graphs have gained some attention[2,12,14,15,16,18,19] in the area of interconnection networks, while the "newer" (n,k)-star graphs have not. In particular, Cheng and Lipman[7] showed that the arrangement graphs can also be oriented so that the resulting directed graphs have the highest possible arc-connectivity; this reinforces the possibility of using arrangement graphs as a replacement for star graphs. Chiang and Chen[11] showed that, with respect to the cost (cost equals diameter $\times$ degree of a vertex), the (n,k)-star graph is better than the arrangement graph $A_{n,k}$. We believe that the arrangement graphs $A_{n,k}$ and the (n,k)-star graphs $S_{n,k}$ are not in competition but complement each other in building a large class of good interconnection networks. For example, the diameter of $A_{n,k}$ depends on k only but the degree depends on both n and k, whereas the degree of $S_{n,k}$ depends on k only but the diameter depends on both n and k. Hence the (n,k)-star graphs deserve some attention as well. In this paper, we show that the more recent (n,k)-star graphs also have the desired property that they can be oriented so that the resulting directed graphs have the highest possible arc-connectivity.

2. Preliminaries

Basic terminology in graph theory can be found in Chartrand and Oellermann[4], and West[22]. Here, we record the basic terminology used in this paper for easy reference.

Notations and Conventions: A graph (respectively directed graph) may contain multiple edges (respectively arcs) but no loops (respectively directed loops). Let H be a graph and X be a proper nonempty subset of the vertex-set. Define $d_H(X)$ to be the number of edges with exactly one end in X. Let G be a directed graph and X be a proper nonempty subset of the vertex-set. Define $\delta_G(X)$ (respectively $\rho_G(X)$) to be the number of arcs leaving (respectively entering) X, that is, the number of arcs with head (respectively tail) in $\overline{X}$ and tail (respectively head) in X. Observe that $\delta_G(X) = \rho_G(\overline{X})$.

Basic Terminology: A graph H is k-edge-connected if the deletion of any $k-1$ edges will not disconnect the graph. This is equivalent to $d_H(X) \geq k$ for every $\emptyset \neq X \subset V$. A directed graph G is *k-arc-connected* if the deletion of any $k-1$ arcs will not disconnect the directed graph. This is equivalent to $\delta_G(X) \geq k$ for every $\emptyset \neq X \subset V$. A graph H is *r-regular* if the degree is r for every vertex v of H. A directed graph G is *r-regular* if the in-degree and out-degree is r, that is, $\delta_G(\{v\}) = \rho_G(\{v\}) = r$ for every vertex v of G. An r-regular undirected graph is *maximally edge-connected* if it is r-edge-connected. An r-regular directed graph is *maximally arc-connected* if it is r-arc-connected. A graph $H = (V, E)$ is *r-connected* if $|V| \geq r+1$ and deleting any set of less than r vertices results in a connected graph. A graph has *connectivity* r if it is r-connected but not $(r+1)$-connected. An r-regular r-connected graph is maximally connected.

An (n, k)-star graph $S_{n,k}$ with $1 \leq k < n$ is governed by the two parameters n and k. The vertex-set of $S_{n,k}$ consists of all the permutations of k elements chosen from the ground set $\{1, 2, \ldots, n\}$. Two vertices $[a_1, a_2, \ldots, a_k]$ and $[b_1, b_2, \ldots, b_k]$ are adjacent if one of the following holds:

1. There exists a $2 \leq r \leq k$ such that $a_1 = b_r$, $a_r = b_1$ and $a_i = b_i$ for $i \in \{1, 2, \ldots, k\} \setminus \{1, r\}$.
2. $a_i = b_i$ for $i \in \{2, \ldots, k\}$, $a_1 \neq b_1$.

Hence given a vertex $[a_1, a_2, \ldots, a_k]$, it has $k-1$ neighbours via the adjacency rule 1 by exchanging a_1 with each of a_i, $i \in \{2, 3, \ldots, k\}$, and it has $n-k$ neighbours via the adjacency rule 2 by exchanging a_1 with each element in $\{1, 2, \ldots, n\} \setminus \{a_1, a_2, a_3, \ldots, a_k\}$. We note that adjacency rule 1 is precisely the rule for the k-dimensional star graph. Hence we will refer the adjacency rule 1 as *star-exchange*. In keeping with the terminology for star graphs, an edge corresponding to a star-exchange, a *star edge*, will be called an *i-edge* if the exchange is between position 1 and position i where $i \in \{2, 3, \ldots, k\}$. We will refer to adjacency rule 2 as *residual-exchange* and an edge corresponding to such an exchange as a *residual-edge*. Figure 1 gives $S_{4,2}$. (We note that for convenience, we write the (n, k)-permutation $[i, j]$ as ij for example, $[1, 4]$ as 14.) Note that given an edge in $S_{n,k}$ with the labellings of its two end-vertices, one can immediately determine whether it is a star-edge or a residual-edge. The family of $S_{n,k}$ generalizes the star graph since $S_{n,n-1}$ is the star graph S_n (an $(n-1)$-permutation on an n-set is really a permutation on n elements). For $S_{n,n-1}$, that is, the star graph S_n, the unique residual-edge for each vertex is its *n-edge*.

The next result contains elementary properties of (n, k)-star graphs. The proofs of these properties are either obvious or can be found in Chiang and Chen[11].

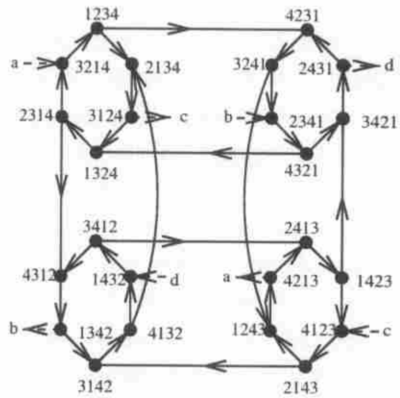

Fig. 1. $S_{4,2}$

Theorem 1 *The (n, k)-star graph $S_{n,k}$ has $n!/(n-k)!$ vertices and is a regular graph with degree $n-1$. Moreover,*

1. *$S_{n,n-1}$ is isomorphic to the star graph S_n for $n \geq 3$.*
2. *$S_{n,1}$ is isomorphic to K_n, the complete graph on n vertices.*
3. *Let $\{x_1, x_2, \ldots, x_k\} \subseteq \{1, 2, \ldots, n\}$ with $k \geq 3$. Let G be the subgraph of $S_{n,k}$ induced by vertices whose labellings are permutations of $x_1, x_2, \ldots, x_k$. Then G is isomorphic to the star graph S_k.*
4. *Let $\{x_2, x_3, \ldots, x_k\} \subseteq \{1, 2, \ldots, n\}$. Let G be the subgraph of $S_{n,k}$ induced by vertices of the form $[y_1, x_2, x_3, \ldots, x_k]$ where $y_1 \in \{1, 2, \ldots, n\} \backslash \{x_2, x_3, \ldots, x_k\}$. Then G is isomorphic to K_{n-k+1}, the complete graph on $n-k+1$ vertices.*
5. *Let G be a subgraph of $S_{n,k}$ with $k \geq 2$ induced by vertices with labellings having the same symbol in the kth position. Then G is isomorphic to $S_{n-1,k-1}$.*
6. *$S_{n,k}$ is vertex-transitive but in general not edge-transitive.*
7. *$S_{n,k}$ is maximally connected (and hence maximally edge-connected).*
8. *$S_{n,k}$ has diameter $2k-1$ if $1 \leq k \leq \lfloor n/2 \rfloor$, and has diameter $k + \lfloor (n-1)/2 \rfloor$ if $\lfloor n/2 \rfloor + 1 \leq k \leq n-1$.*

A star subgraph of $S_{n,k}$ using the rule in (3) of Theorem 1 will be called a *fundamental star*. A complete subgraph of $S_{n,k}$ using the rule in (4) of Theorem 1 will be called a *fundamental clique*. It is clear that there are $\binom{n}{k}$ fundamental stars and $\binom{n}{k-1}(k-1)! = \frac{n!}{(n-k+1)!}$ fundamental cliques.

Like other interconnection networks, $S_{n,k}$ may contain exponentially many vertices (with respect to n and k). Hence we cannot efficiently apply regular shortest path algorithms. One needs an algorithm polynomial in n and k or perhaps even a distributed routing algorithm. A distributed routing from a to b means that at an intermediate vertex v, the next step is determined by v and b only and does not require information about the previous steps nor impose any future steps. In the paper by Chiang and Chen[11], a simple and efficient optimal (in terms of path length) distributed routing algorithm is given to find the shortest path between any two vertices of $S_{n,k}$. Our objective is to orient $S_{n,k}$ in such a way that the resulting directed graph is $\frac{n-1}{2}$-regular and $\frac{n-1}{2}$-arc-connected when $n-1$ is even. Of course such an orientation exists from a theorem of Nash-Williams[21] as $S_{n,k}$ is $(n-1)$-regular and $(n-1)$-edge-connected. (In fact, if $n-1$ is even then the proof of this theorem is easy as well, because the graph is Eulerian.) But here, we also want the resulting directed graph to have a small diameter and to obtain an efficient optimal or near-optimal routing algorithm. In fact, we will like to have a local orientation rule, that is, given an edge in $S_{n,k}$, its orientation is determined only by its end-vertices. If $n-1$ is odd, we will show that $S_{n,k}$ (except for certain degenerate cases) can be made n-regular and n-edge-connected by adding a carefully chosen perfect matching from its complement; moreover, we solve its orientation problem.

3. Ingredient I: More than maximally-edge-connected

In this section, we show that, except for certain degenerate cases, the (n,k)-star graph is "more" than maximally-edge-connected. We now state some necessary definitions. An undirected noncomplete r-regular graph is *loosely super connected* or simply *super connected* if its only minimum disconnecting vertex-sets are those induced by the neighbours of a vertex. This is a much stronger property than requiring the connectivity to be r. If, in addition, the deletion of a minimum disconnecting set always results in a graph with exactly two components, one of which has only one vertex, then the graph is *tightly super connected.* Note that the complete bipartite graph $K_{r,r}$ with $r \geq 3$ is loosely super connected and not tightly super connected. We note that we exclude complete graphs in our discussion. This is not a restriction as a complete graph is too expensive to be used as an interconnection network. Another related notion is that of *super edge-connectedness.* A graph is *super edge-connected* if the only minimum edge-disconnecting sets are those induced by a vertex. (Obviously, the notion of "loosely superness" and "tightly superness" are the same in this case.) The notion of "superness" was first introduced by Bauer, Boesch, Suffel and Tindell[3].

Our goal is to show that $S_{n,k}$ is tightly super connected. Since $S_{n,1}$ is the complete graph on n vertices, we ignore this trivial case. Consider $k=2$. Since $S_{3,2}$ is a 6-cycle, it is not super connected. From Figure 1, we can see that $S_{4,2}$ is not super connected as deleting the three vertices $12, 13, 14$ gives a graph with two nontrivial components. This can easily be generalized to $S_{n,2}$; hence $S_{n,2}$ is not super connected for $n \geq 3$. However, our next result shows that $S_{n,k}$ is tightly super connected if $k \geq 3$.

Theorem 2 *$S_{n,2}$ is not super connected for $n \geq 3$. If $n > k \geq 3$, then $S_{n,k}$ is tightly super connected.*

Proof. We have already seen that $S_{n,2}$ is not super connected for $n \geq 3$. Suppose $k \geq 3$ and $n > k$. Let H_i be the subgraph of $S_{n,k}$ with i in the last position for $1 \leq i \leq n$. Then by Theorem 1, H_i is isomorphic to $S_{n-1,k-1}$. It is clear that every vertex in H_i has exactly one neighbour not in H_i. We also note that there are $(n-2)!/(n-k)!$ independent[1] edges between H_i and H_j for $1 \leq i < j \leq n$. Let T be a set of vertices in $S_{n,k}$ such that $|T| = n-1$. Assume $S_{n,k} \setminus T$ is disconnected. We want to show that T is the set of neighbours of a unique vertex, and that $S_{n,k} \setminus T$ has exactly two components. Let $T_i = V(H_i) \cap T$ and $t_i = |T_i|$ for $1 \leq i \leq n$. So $\sum_{i=1}^{n} t_i = n-1$.

Case 1: $|t_i| \leq n-3$ for $1 \leq i \leq n$. Then $H_i \setminus T_i$ is connected since $S_{n-1,k-1}$ is $(n-2)$-connected by Theorem 1. There are $(n-2)!/(n-k)!$ independent edges between H_i and H_j, $t_i \leq n-3$ and $t_j \leq n-3$. Hence there are at least $\alpha_{i,j} = (n-2)!/(n-k)! - \min\{t_i + t_j, n-1\}$ edges between $H_i \setminus T_i$ and $H_j \setminus T_j$.

Case 1a: $k \geq 4$. Then $\alpha_{i,j} \geq (n-2)(n-3) - (n-1) \geq 1$ as $n > k \geq 4$. Hence there are edges between $H_i \setminus T_i$ and $H_j \setminus T_j$. Therefore, $S_{n,k} \setminus T$ is still connected, a contradiction.

Case 1b: $k = 3$. Note that $\alpha_{i,j} = (n-2) - \min\{t_i + t_j, n-1\}$. If $\alpha_{i,j} \geq 1$ for all $1 \leq i < j \leq n$, then we are done, as before. If this is not the case, then there is a set $\{i, j\}$ such that $\alpha_{i,j} \leq 0$. For notational convenience, we assume $i = 1$ and $j = 2$. Hence $t_1 + t_2 = n-1$ or $t_1 + t_2 = n-2$. Hence $t_3 + t_4 + \cdots + t_n \leq 1$, that is, $T_3 \cup T_4 \cup \cdots \cup T_n$ has at most one vertex. Clearly, $S_{n,k} \setminus (V(H_1) \cup V(H_2) \cup T_3 \cup T_4 \cup \cdots \cup T_n)$ is connected. Since $t_1 \leq n-3$, $t_3 + t_4 + \cdots + t_n \leq 1$ and $n \geq 4$, either $\alpha_{1,3} \geq 1$ or $\alpha_{1,4} \geq 1$. Hence there is an edge between $H_1 \setminus T_1$ and $S_{n,k} \setminus (V(H_1) \cup V(H_2) \cup T_3 \cup T_4 \cup \cdots \cup T_n)$. Similarly, there is an edge between $H_2 \setminus T_2$ and $S_{n,k} \setminus (V(H_1) \cup V(H_2) \cup T_3 \cup T_4 \cup \cdots \cup T_n)$. Hence $S_{n,k} \setminus T$ is connected, a contradiction.

Case 2: $t_i > n-3$ for some i. Without lost of generality, we may assume t_1 is the maximum among the t_i's for notational convenience. The remaining cases are $t_1 = n-1$ and $t_1 = n-2$. We note that $H_1 \setminus T_1$ may be disconnected. If $t_1 = n-1$, then $T_i = \emptyset$ for $2 \leq i \leq n$. Since every vertex in $H_1 \setminus T_1$ is adjacent to a vertex not in H_1 and this vertex is not in T, $S_{n,k} \setminus T$ is connected, a contradiction. We now assume $t_1 = n-2$. Let x be the unique vertex in $T \setminus T_1$. Clearly, $S_{n,k} \setminus (V(H_1) \cup \{x\})$ is connected. If $H_1 \setminus T_1$ is connected, then $S_{n,k} \setminus T$ is connected (since $H_1 \setminus T_1$ has at least two vertices), a contradiction. Assume $H_1 \setminus T_1$ is not connected. Consider a component C in $H_1 \setminus T_1$. If C has two distinct vertices say u and v, let y be the neighbour of u that is not in H_1 and z be the neighbour of v that is not in H_1. Since $y \neq z$, at least one of y and z is not x; hence the vertices in C and the vertices of $S_{n,k} \setminus (V(H_1) \cup \{x\})$ belong to the same component in $S_{n,k} \setminus T$. If C has only one vertex say w, then w and the vertices of $S_{n,k} \setminus (V(H_1) \cup \{x\})$ belong to the

[1] A set of edges are *independent* if no two of them are incident to the same vertex.

same component in $S_{n,k} \setminus T$ unless x is a neighbour of w; in this case, T is the set of neighbours of w. Since x has only one neighbour in H_1, we can conclude that $S_{n,k} \setminus T$ has exactly two components, one of which has only one vertex. Hence $S_{n,k}$ is tightly super connected. □

As a corollary of Theorem 2, we have the next result whose proof is given by Cheng and Lipman[7]. In fact, the proof of Theorem 2 given here is a slight modification of the proof given Cheng and Lipman[7] for Corollary 3.

Corollary 3 *S_3 is not super connected. The star graph S_n is tightly super connected for $n \geq 4$.*

The proof of the next result is given in Cheng and Lipman[5], and Cheng, Lipman and Park[9] so we will state it without proof.

Theorem 4 [2]*Let $H = (V,E)$ be a κ-regular tightly super connected graph with $\kappa \geq 1$. If H has more than $2\kappa + 2$ vertices, then it is super edge-connected.*

Corollary 5 *Let $k \geq 3$. Then $S_{n,k}$ is super edge-connected.*

Proof. This follows from Theorem 2 and Theorem 4 as $S_{n,k}$ has more than $2(n-1)+2$ vertices since $n > k \geq 3$. □

Corollary 5 will be useful in Section 6.

4. Ingredient II: Orientation of substructures

Recall that $S_{n,k}$ contains $\binom{n}{k}$ fundamental stars and $\binom{n}{k-1}(k-1)! = \frac{n!}{(n-k+1)!}$ fundamental cliques. Moreover, these fundamental stars and fundamental cliques partition the edge-set of $S_{n,k}$. Furthermore, every vertex is on exactly one fundamental star and exactly one fundamental clique. Since our goal is to find a good orientation for $S_{n,k}$, it is useful to find a good orientation for complete graphs and star graphs.

Proposition 6 *Let $q \geq 3$ be odd. Then K_q can be oriented into a $\frac{q-1}{2}$-regular directed graph such that every arc is on a directed 3-cycle.*

Proof. Assume the labels of the q vertices are $1, 2, \ldots, q$. Given an edge between i and j where $i < j$, we orient it from i to j if i and j have different parity, and orient it from j to i if i and j have the same parity. Then it is easy to see that the resulting directed graph is $\frac{q-1}{2}$-regular. Now consider an arc $a \to b$.

1. Suppose $b < a$. Then a and b have the same parity and hence $a \geq b+2$. We have the following directed 3-cycle: $a \to b$, $b \to (b+1)$ and $(b+1) \to a$.

2. Suppose $a < b$. Then a and b have different parity. If $a \neq 1$, then we have the following directed 3-cycle: $a \to b$, $b \to (a-1)$ and $(a-1) \to a$. If $a = 1$, then $b \neq q$ since q is odd. Hence we have the following directed 3-cycle: $a \to b$, $b \to (b+1)$ and $(b+1) \to a$. □

[2] This is a weaker version of a more general result. For our purpose, this weaker version is enough. See Cheng, Lipman and Park[8] for the statement and a proof of the stronger result.

Although the proof of the next result is almost identical to that of Proposition 6, we include it here as we need the orientation explicitly given in the proof.

Proposition 7 *Let $q \geq 4$ be even. Then K_q can be oriented into a directed graph such that half of the vertices have in-degree $\lfloor \frac{q-1}{2} \rfloor$ and half of the vertices have in-degree $\lceil \frac{q-1}{2} \rceil$. Moreover, every arc is on a directed 3-cycle or a directed 4-cycle.*

Proof. Assume the labels of the q vertices are $1, 2, \ldots, q$. Given an edge between i and j where $i < j$, we orient it from i to j if i and j have different parity, and orient it from j to i if i and j have the same parity. Then it is easy to see that the resulting directed graph has the required degree property. To be precise, a vertex with an odd label has out-degree $\lceil \frac{q-1}{2} \rceil$ and in-degree $\lfloor \frac{q-1}{2} \rfloor$; a vertex with an even label has in-degree $\lceil \frac{q-1}{2} \rceil$ and out-degree $\lfloor \frac{q-1}{2} \rfloor$. Now consider an arc $a \to b$.

1. Suppose $b < a$. Then a and b have the same parity and hence $a \geq b + 2$. We have the following directed 3-cycle: $a \to b$, $b \to (b+1)$ and $(b+1) \to a$.

2. Suppose $a < b$. Then a and b have different parity. If $a \neq 1$, then we have the following directed 3-cycle: $a \to b$, $b \to (a-1)$ and $(a-1) \to a$. If $b \neq q$, then we have the following directed 3-cycle: $a \to b$, $b \to (b+1)$ and $(b+1) \to a$. The remaining case is $a = 1$ and $b = q$. Since $q \geq 4$, the following vertices are distinct: $1, 2, 3, q$ and we have the following directed 4-cycle: $1 \to q$, $q \to 2$, $2 \to 3$ and $3 \to 1$. □

Our goal is to orient the fundamental stars and fundamental cliques separately so that the resulting directed (n, k)-star graph has the desired property. In particular, we would like the resulting directed graph to be regular. For example, if each fundamental star and fundamental clique can be oriented as a regular graph, then the resulting graph is regular. But this is not always possible. So we desire a good terminology to indicate when an orientation is regular, and when it is not, we want a descriptive name. So it is reasonable to call the orientation given in the proof of Proposition 6 a *balanced orientation.* The term is to indicate the resulting orientation is regular. The orientation given in the proof of Proposition 7 can be called an *oDD-more-out orientation.* (The term is to indicate an odd labelled vertex has a larger out-degree in the orientation. We purposely use "oDD" instead of "odd" as a different kind of oddness will be introduced later on.) If we reverse the rule, such an orientation can be called an *oDD-more-in orientation.* The orientation results given in Proposition 6 and Proposition 7 will be useful later on.

Remark 8 *If we replace the labellings of the vertices, namely, $1, 2, 3, \ldots, q$, by the labellings $2, 3, 4, \ldots, q+1$, in the proof of Proposition 6, the resulting orientation is still balanced.*

We now describe the orientation of star graphs given by Day and Tripathi[17]. Let π_a and π_b be adjacent in S_n through an i-edge. We may assume π_a is even[3]and π_b is odd. Then the edge is oriented from π_a to π_b if i is even, and the edge is oriented from π_b to π_a if i is odd. The resulting graph is denoted by US_n. Figure 2 gives US_4. Let π be a vertex in US_n. Then its in-degree and out-degree are $\lceil\frac{n-1}{2}\rceil$ and $\lfloor\frac{n-1}{2}\rfloor$ respectively if π is odd and vice versa if π is even. This orientation of the star graph S_n will be called the *Day-Tripathi orientation.* If n is odd, then the resulting directed graph is regular; hence the Day-Tripathi orientation is a *balanced orientation* if n is odd. If n is even, then a vertex corresponding to an odd permutation has in-degree one more than the out-degree; hence we say the Day-Tripathi orientation is an *ODd-more-in orientation* when n is even. (Note that we purposely use "ODd" instead of "oDD" or "odd" as "ODd" here indicate odd permutations.) The next result is easy to see and will be used in subsequent sections.

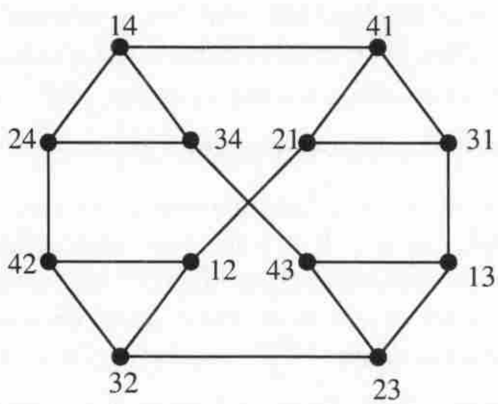

Fig. 2. US_4: odd-more-in

Proposition 9 *Every arc in* US_n *with* $n \geq 3$ *is on a directed 6-cycle.*

5. Global oddness, global evenness and local orientation

In the previous section, we use terms such as *balanced orientation, oDD-more-in orientation* and *ODd-more-in orientation* to describe certain orientations on star graphs and complete graphs. The idea in orienting $S_{n,k}$ is to orient the fundamental stars and the fundamental cliques by using such rules. However, we have seen the dual usage of the word *odd* to describe a vertex. The word *odd* corresponds to an odd permutation in S_n whereas the word *odd* corresponds to an odd numbered label in the case of a complete graph. Hence we have used "oDD" and "ODd" instead of a generic "odd" in terms such as *odd-more-in orientation.* In this section, we remedy this by defining oddness and evenness for the vertices in $S_{n,k}$. Note also that since

[3]A permutation is *even* (*odd*) if it can be written as a product of an even (odd) number of transpositions.

the orientation rule for a complete graph depends on the ordering of the vertices, we need to have a rule to address this using the labellings of vertices in $S_{n,k}$. Finally, we present a local orientation rule, that is, given an edge, an orientation is determined completely by the labels of the two end-vertices.

Given an (n, k)-star graph $S_{n,k}$, we map each vertex to a unique full permutation of $\{1, 2, \dots, n\}$. Suppose a vertex in $S_{n,k}$ has the labelling $[a_1, a_2, \cdots a_k]$. Then the unique permutation on $\{1, 2, \dots, n\}$ associated with it is $[a_1, a_2, \cdots, a_k, x_1, \cdots, x_{n-k}]$ where $\{x_1, x_2, \dots, x_{n-k}\} = \{1, 2, \dots, n\} \setminus \{a_1, a_2 \dots, a_k\}$ and $x_1 < x_2 < \cdots < x_{n-k}$. We call a vertex *odd* if its associated permutation is odd and *even* if its associated permutation is even. We note that under this definition, a star-exchange still induces an edge between an odd vertex and an even vertex. This definition is consistent with the basic properties and terminology of star graphs. We want to give a local orientation rule to the edges so that the following is true:

1. If k is odd, then every fundamental star has a *balanced orientation* with regularity $\frac{k-1}{2}$; if k is even, then every fundamental star has an *odd-more-in orientation* with every odd vertex having in-degree $\left\lceil \frac{k-1}{2} \right\rceil$ and out-degree $\left\lfloor \frac{k-1}{2} \right\rfloor$, and every even vertex having out-degree $\left\lceil \frac{k-1}{2} \right\rceil$ and in-degree $\left\lfloor \frac{k-1}{2} \right\rfloor$.

2. If $n - k + 1$ is odd, then every fundamental clique has a *balanced orientation* with regularity $\frac{n-k}{2}$. If $n - k + 1$ is even, then either every fundamental star has an *odd-more-out orientation* or every fundamental star has an *odd-more-in orientation*. In an *odd-more-out orientation*, every odd vertex having out-degree $\left\lceil \frac{n-k}{2} \right\rceil$ and in-degree $\left\lfloor \frac{n-k}{2} \right\rfloor$, and every even vertex having in-degree $\left\lceil \frac{n-k}{2} \right\rceil$ and out-degree $\left\lfloor \frac{n-k}{2} \right\rfloor$. In an *odd-more-in orientation*, every odd vertex having in-degree $\left\lceil \frac{n-k}{2} \right\rceil$ and out-degree $\left\lfloor \frac{n-k}{2} \right\rfloor$, and every even vertex having out-degree $\left\lceil \frac{n-k}{2} \right\rceil$ and in-degree $\left\lfloor \frac{n-k}{2} \right\rfloor$.

If we can accomplish this, then the terminology odd-more-in orientation and odd-more-out orientation may replace oDD-more-in (out) for fundamental cliques and ODd-more-in (out) for fundamental stars. For the rest of this section, we assume $n - k \geq 2$ as we need Propositions 6 and 7.

Recall that given an edge in $S_{n,k}$ with the labellings of its two end-vertices, one can immediately determine whether it is a star-edge or a residual-edge. Consider an i-edge with end-vertices π_a and π_b. We may assume π_a is even and π_b is odd. Then the edge is oriented from π_a to π_b if i is even, and the edge is oriented from π_b to π_a if i is odd following the Day-Tripathi rule. This gives the orientation of the fundamental stars.

To give a local orientation rule for a fundamental clique, we note that given a residual-edge, one can immediately determine the vertices of the fundamental clique containing this edge. Let the vertices of this fundamental clique be

$$[x_1, a_2, \dots, a_k], [x_2, a_2, \dots, a_k], \dots, [x_{n-k+1}, a_2, \dots, a_k]$$

with $x_1 < x_2 < x_3 < \ldots < x_{n-k+1}$. Next we note that the parity of the above vertices (with respect to their associated permutations) alternates. At first glance, we may want to copy the rule from the previous section as follows: Given the two end-vertices of a residual-edge, say $[x, a_2, \ldots, a_k]$ and $[y, a_2, \ldots, a_k]$. Suppose $x < y$. Then we orient it from $[x, a_2, \ldots, a_k]$ to $[y, a_2, \ldots, a_k]$ if $[x, a_2, \ldots, a_k]$ and $[y, a_2, \ldots, a_k]$ have different parity, and orient it from $[y, a_2, \ldots, a_k]$ to $[x, a_2, \ldots, a_k]$ if $[x, a_2, \ldots, a_k]$ and $[y, a_2, \ldots, a_k]$ have the same parity. In other words, we map

$$[x_1, a_2, \ldots, a_k], [x_2, a_2, \ldots, a_k], \ldots, [x_{n-k+1}, a_2, \ldots, a_k]$$

to $1, 2, 3, \ldots, n-k+1$ as an ordered list and apply the rules given in Propositions 6 and 7 as $n - k \geq 2$. Note that this has a small problem as $[x_1, a_2, \ldots, a_k]$ may not be odd. However, the resulting orientation for this fundamental star is still balanced if $n - k + 1$ is odd as indicated by Remark 8. Suppose $n - k + 1$ is even and $[x_1, a_2, \ldots, a_k]$ is even. Then $[x_{n-k+1}, a_2, \ldots, a_k]$ is odd. So we map

$$[x_1, a_2, \ldots, a_k], [x_2, a_2, \ldots, a_k], \ldots, [x_{n-k+1}, a_2, \ldots, a_k]$$

to $n - k + 1, n - k, \ldots, 3, 2, 1$ as an ordered list and apply the rules given in Proposition 7 as $n - k \geq 2$. In other words, suppose the two end-vertices of a residual-edge are $[x, a_2, \ldots, a_k]$ and $[y, a_2, \ldots, a_k]$. Suppose $x > y$. Then we orient it from $[x, a_2, \ldots, a_k]$ to $[y, a_2, \ldots, a_k]$ if $[x, a_2, \ldots, a_k]$ and $[y, a_2, \ldots, a_k]$ have different parity, and orient it from $[y, a_2, \ldots, a_k]$ to $[x, a_2, \ldots, a_k]$ if $[x, a_2, \ldots, a_k]$ and $[y, a_2, \ldots, a_k]$ have the same parity. Then the resulting orientation is odd-more-out. Of course, if we reverse the rule, it is odd-more-in.

We note that such a rule is a local orientation rule. For example, suppose our intention is to give every fundamental clique an odd-more-out orientation. Consider the residual-edge between $[9, 1, 6, 5, 4]$ and $[3, 1, 6, 5, 4]$ in $S_{10,5}$. From these labellings, we know this residual-edge belongs to the fundamental clique with the following vertex-set:

$$[2, 1, 6, 5, 4], [3, 1, 6, 5, 4], [7, 1, 6, 5, 4], [8, 1, 6, 5, 4], [9, 1, 6, 5, 4], [10, 1, 6, 5, 4].$$

Now the associated permutation for $[2, 1, 6, 5, 4]$ is $[2, 1, 6, 5, 4, 3, 7, 8, 9, 10]$ which is an odd permutation. Hence $[2, 1, 6, 5, 4]$ is odd in $S_{10,5}$. Since $3 < 9$, and the two vertices $[9, 1, 6, 5, 4]$ and $[3, 1, 6, 5, 4]$ have different parity, we orient the edge from $[3, 1, 6, 5, 4]$ to $[9, 1, 6, 5, 4]$. Now consider the residual-edge between $[9, 1, 6, 4, 5]$ and $[3, 1, 6, 4, 5]$ in $S_{10,5}$. From these labellings, we know this residual-edge belongs to the fundamental clique with the following vertex-set:

$$[2, 1, 6, 4, 5], [3, 1, 6, 4, 5], [7, 1, 6, 4, 5], [8, 1, 6, 4, 5], [9, 1, 6, 4, 5], [10, 1, 6, 4, 5].$$

Now the associated permutation for $[2, 1, 6, 4, 5]$ is $[2, 1, 6, 4, 5, 3, 7, 8, 9, 10]$ which is an even permutation. Hence $[2, 1, 6, 4, 5]$ is even in $S_{10,5}$. Since $3 < 9$, and the two vertices $[9, 1, 6, 4, 5]$ and $[3, 1, 6, 4, 5]$ have different parity, we orient the edge from $[9, 1, 6, 4, 5]$ to $[3, 1, 6, 4, 5]$.

6. Orientation of $S_{n,k}$

Our objective is to give an orientation to $S_{n,k}$ so that the resulting directed graph is highly connected and has a small diameter; moreover, we want a good routing algorithm. Throughout the section, we assume $k \geq 3$. Since the case for star graphs has been studied already, we assume $n - k \geq 2$ throughout the section. We begin with the following simple result.

Proposition 10 *Let $H = (V, E)$ be a $(2r)$-regular $(2r)$-edge-connected graph. Let G be an orientation of H. If G is r-regular, then G is r-arc-connected.*

Proof. Let X be a nonempty proper subset of V. Then $\sum_{v \in X} \delta_G(\{v\}) = \gamma_G(X) + \delta_G(X)$ and $\sum_{v \in X} \rho_G(\{v\}) = \gamma_G(X) + \rho_G(X)$ where $\gamma_G(X)$ is equal to the number of arcs with both head and tail in X. Since G is r-regular, $\sum_{v \in X} \delta_G(\{v\}) = \sum_{v \in X} \rho(\{v\}) = r|X|$; hence $\delta_G(X) = \rho_G(X)$. Therefore, $d_H(X) = \delta_G(X) + \rho_G(X) = 2\delta_G(X)$. Since H is $(2r)$-edge-connected, $d_H(X) \geq 2r$. Hence $\delta_G(X) \geq r$ and we are done. □

This is of immediate interest to us since for connectivity purpose, we only need to orient $S_{n,k}$ such that the resulting directed graph is regular (if possible). Suppose $n-1$ is even. Since $S_{n,k}$ is maximally-edge-connected, our goal is to orient $S_{n,k}$ into a regular graph. We consider two cases: $k-1$ is even, and $k-1$ is odd. Since $n-1$ is even, $k-1$ being even is equivalent to $n-k$ being even and $k-1$ being odd is equivalent to $n-k$ being odd.

Suppose $k \geq 3$, $k-1$ is even (and hence $n-k \geq 2$ is even). Hence k is odd and $n-k+1 \geq 3$ is odd. Therefore, we give the fundamental stars *balanced orientations* and we give the fundamental cliques *balanced orientations.* In this case, let the resulting directed graph be $\overrightarrow{S_{n,k}}$, and we have the following result.

Theorem 11 *Let $n-1$ be even, $k \geq 3$ and $n-k \geq 2$ be even. Then $\overrightarrow{S_{n,k}}$ is a $\frac{n-1}{2}$-regular directed graph such that every arc is on a directed 3-cycle or a directed 6-cycle.*

Proof. The regularity condition is clear. If an arc is in an oriented residual-edge, then it is on directed 3-cycle by Proposition 6. If the arc is an oriented star-edge, then it is an arc in a subgraph isomorphic to US_k. Hence by Proposition 9 as $k \geq 3$, it is on a directed 6-cycle. □

Now suppose $n-k$ is odd and $k \geq 3$. Then $n-k+1 \geq 4$ is even and k is even. Then we give every fundamental star an odd-more-in orientation and every fundamental clique an odd-more-out orientation. Hence such an orientation gives a $\frac{n-1}{2}$-regular directed graph. This orientation is denoted by $\overrightarrow{S_{n,k}}$ in this case. This together with Propositions 7 and 9 give the next result.

Theorem 12 *Let $n-1$ be even, $k \geq 3$ and $n-k \geq 2$ be odd. Then $\overrightarrow{S_{n,k}}$ is a $\frac{n-1}{2}$-regular directed graph such that every arc is on a directed 3-cycle, a directed 4-cycle or a directed 6-cycle.*

A regular directed graph G has many advantages such as $\delta_G(X) = \rho_G(X)$ for every nonempty proper subset X of the vertex-set. Since regularity is important in the design of interconnection networks, it is reasonable to insist that an unidirectional interconnection network to be regular. If $n-1$ is odd, then it is impossible to orient $S_{n,k}$ into a regular directed graph. However, we will show that it is possible to orient $S_{n,k}$ with the property that half of the vertices have in-degree $\lceil \frac{n-1}{2} \rceil$ and half of the vertices have in-degree $\lfloor \frac{n-1}{2} \rfloor$. Hence if we add a directed perfect matching[4] of size $\frac{n!}{2(n-k)!}$ between these two sets (with the proper directions), the augmented directed graph is regular.

Proposition 13 *Let $k \geq 3$. Then exactly half of the vertices in $S_{n,k}$ is even.*

Proof. Since every vertex appears in a unique fundamental star and exactly half of the vertices in every fundamental star is even, the result follows. □

Suppose $n-1$ is odd, $k \geq 3$ and $n-k \geq 2$. We consider two cases. Suppose k is odd. Then $n-k+1 \geq 3$ is even. We give every fundamental star a balanced orientation and every fundamental clique an odd-more-in orientation. Suppose k is even. Then $n-k+1 \geq 3$ is odd. Then we give every fundamental star an odd-more-in orientation and every fundamental clique a balanced orientation. In either case, the resulting directed graph is denoted by $\overrightarrow{S_{n,k}}$.

Proposition 14 *Let $n-1$ be odd, $k \geq 3$ and $n-k \geq 2$. Then $\overrightarrow{S_{n,k}}$ is a directed graph with odd vertices having in-degree $\left\lceil \frac{n-1}{2} \right\rceil$ and out-degree $\left\lfloor \frac{n-1}{2} \right\rfloor$, and even vertices having in-degree $\left\lfloor \frac{n-1}{2} \right\rfloor$ and out-degree $\left\lceil \frac{n-1}{2} \right\rceil$.*

Let $n-1$ be odd, $k \geq 3$ and $n-k \geq 2$. Let Z be the set of odd vertices and Y be the set of even vertices. If a perfect matching of size $\frac{n!}{2(n-k)!}$ between Y and Z is added to $\overrightarrow{S_{n,k}}$ and with orientation from Z to Y, the resulting graph is $\lceil \frac{n-1}{2} \rceil$-regular. There are many choices of such a directed perfect matching. We would like to find one so that the resulting graph is maximally arc-connected. It turns out that any such perfect matching will work; we will use one that is convenient.

Theorem 15 *Let $n-1$ be odd, $k \geq 3$ and $n-k \geq 2$. Let M be a perfect matching between the set of even vertices and the set of odd vertices. Then the graph (with multiple edges allowed) obtained by adding M to $S_{n,k}$ is n-regular and n-edge-connected.*

Proof. Let H be the resulting graph. It is clear that H is n-regular. Let X be a nonempty proper subset of the vertex set. Suppose X is not a singleton. Then $d_H(X) \geq n$ since $d_{S_{n,k}}(X) \geq n$ as $S_{n,k}$ is a $(n-1)$-regular super edge-connected graph by Corollary 5. If X is a singleton, then $d_H(X) = n$ as H is n-regular. Hence H is n-edge-connected. □

[4]Given a graph $H = (V, E)$, $M \subseteq E$ is a *perfect matching* if every vertex of the graph $H' = (V, M)$ has degree one. Moreover, M is said to be *between* Y and Z where Y and Z form a partition of the vertex-set if every element of M has exactly one end in Y and one end in Z.

Now $n-1$ be odd, $k \geq 3$ and $n-k \geq 2$. Let $\mathcal{M}$ be any perfect matching in the complement of $S_{n,k}$ between Z (the set of odd vertices) and Y (the set of even vertices) in $\overrightarrow{S_{n,k}}$ and are directed from Z to Y. In particular, we choose the matching consisting of edges of the form $(u, f(u))$ where $f(u)$ is obtained from u by interchanging the second and third symbols in the labelling of u. Clearly the associated permutations of u and $f(u)$ have opposite parities, hence this is indeed a perfect matching between Z and Y. Let the resulting directed graph be $\overrightarrow{AS_{n,k}}$, the augmented oriented (n,k)-star with the direction of the edges in the added matching oriented from Z to Y.

Theorem 16 *Let $n-1$ be odd, $k \geq 3$ and $n-k \geq 2$. Then $\overrightarrow{AS_{n,k}}$ is a $\frac{n}{2}$-regular maximally-arc-connected directed graph. Moreover, every arc is on a directed 3-cycle, a directed 4-cycle or a directed 6-cycle.*

7. Connectivity and routing of unidirectional (n,k)-star graphs

In Section , we augmented the oriented $S_{n,k}$ into a regular directed graph $\overrightarrow{AS_{n,k}}$ if $n-1$ is odd. We will refer to $\overrightarrow{S_{n,k}}$ if $n-1$ is even and $\overrightarrow{AS_{n,k}}$ if $n-1$ is odd as *unidirectional (n,k)-star graphs.* We have already seen that unidirectional (n,k)-star graphs are $\left\lceil \frac{n-1}{2} \right\rceil$-regular maximally-arc-connected directed graphs. In this section, we will consider the diameter and routing properties of unidirectional (n,k)-star graphs.

Note that we can apply any optimal routing algorithm for $S_{n,k}$ to obtain a near optimal routing algorithm for $\overrightarrow{S_{n,k}}$ and $\overrightarrow{AS_{n,k}}$. Given two vertices a and b, we apply a routing algorithm for $S_{n,k}$. In an intermediate step from a to b using the edge (x,y), if it is directed in the right direction in $\overrightarrow{S_{n,k}}$ then use $x \to y$; otherwise, we replace this one-step move by a $(p-1)$-step move using a directed p-cycle where $p \in \{3,4,6\}$ by Theorems 11, 12 and 16. In particular, we can use the distributed routing algorithm given by Chiang and Chen[11]. The length of this directed path from a to b is at most five times the length of the undirected path between a and b. Hence this algorithm produces a directed path of length at most five times the optimal length. This gives a near-optimal solution with the ratio of performance guarantee being 5. We note that it is possible for the length of an optimal routing to grow five times from the undirected case to the directed case. Suppose $k \geq 3$ and $n-k \geq 2$. It is easy to see that a star-edge (x,y) is not on a cycle of size smaller than 6. Suppose in $\overrightarrow{S_{n,k}}$, it is directed from y to x. Then any routing from x to y in $\overrightarrow{S_{n,k}}$ has length at least 5. This, of course, does not imply that the diameter of $\overrightarrow{S_{n,k}}$ is five time that of $S_{n,k}$. Nevertheless, this observation together with Theorem 1 (the diameter of $S_{n,k}$) provides a simple upper bound.

Theorem 17 *Suppose $k \geq 3$ and $n-k \geq 2$. Then in an unidirectional (n,k)-star graph, there is an efficient routing algorithm (in n and k) to find a directed path from a to b whose length is at most five times the distance from a to b. In particular, its diameter is at most $10k-5$ if $1 \leq k \leq \lfloor n/2 \rfloor$, and at most $5k + 5\lfloor (n-1)/2 \rfloor$ if $\lfloor n/2 \rfloor + 1 \leq k \leq n-1$.*

We note that we only use the term efficient rather than giving an explicit running time in Theorem 17 because we have not described the algorithm given by Chiang and Chen[11]. Without knowing the algorithm exactly, one simply examines every edge in the routing produced by the algorithm and checks for its direction to see whether a replacement of a directed path of length two or three or five is necessary. Moreover, such a replacement is readily available as the direction procedures given for fundamental stars and fundamental cliques are explicit and hence such a replacement (if necessary) is also explicit. Hence, any efficient routing algorithm for $S_{n,k}$ will give an efficient routing algorithm for $\overrightarrow{S_{n,k}}$. Indeed an efficient optimal routing algorithm for $S_{n,k}$ is given by Chiang and Chen[11]. Therefore, we have an efficient near-optimal routing algorithm for $\overrightarrow{S_{n,k}}$. This algorithm is not strictly distributed. A distributed routing from x to y means that at an intermediate vertex v, the next step is determined by v and y only and does not require information about the previous steps nor impose any future steps. Now, for our algorithm, we may need to replace a reversed arc $a \leftarrow b$ by a directed path of length two or three or five, hence a imposes future moves. This contradicts our requirement for a distributed routing algorithm. However, this imposition is for the *next* p moves *only* and p is bounded above by a constant 5, so it still retains the characteristics of a distributed routing algorithm. In fact, one can view a step of being p consecutive moves with $p \leq 5$.

Although we have only provided a near-optimal routing algorithm and not an optimal routing algorithm, we note that even for the well-studied special case, the star graph, an optimal routing algorithm for the unidirectional star graph is unknown. However, Day and Tripathi[17] gives a very good near-optimal algorithm for the unidirectional star graph.

8. Concluding Remarks

In this paper, we studied the orientation problem on the generalized (n, k)-star graphs which are generalizations of the popular star graphs. If the regularity of $S_{n,k}$ is even, then we gave an orientation to obtain a regular directed graph $\overrightarrow{S_{n,k}}$. If the regularity of $S_{n,k}$ is odd, then we added a perfect matching to it and gave an orientation of the resulting graph to obtain a regular directed graph $\overrightarrow{AS_{n,k}}$. In either case, the resulting directed graph is maximally-arc-connected. In addition, its diameter is small and we gave an efficient near-optimal routing algorithm. These graphs provide a large hierarchy of classes of unidirectional interconnection networks with good properties. This shows that the unidirectional (n, k)-star graphs are very desirable. An open problem is to find a better estimate of the diameter of these graphs.

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