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Two-dimensional homing sort

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ABSTRACT

Homing sort, i.e., sorting by placement and shift, is a natural way to do hand-sorting. Elizalde and Winkler showed that (1) any n-element permutation can be sorted by $n-1$ or less one-dimensional homing operations; (2) no n-element permutation admits a sequence of 2^{n-1} or more homing operations; and (3) the number of n-element permutations that admit a sequence of $2^{n-1}-1$ homing operations is super-exponential in n. In the present paper, we study sorting via two-dimensional homing operations and obtain the following observations: (1) Any $m \times n$ permutation can be sorted by at most $mn-1$ two-dimensional homing operations. (2) If both vertical-first and horizontal-first homing operations are allowed, for any integers $m \geqslant 2$ and $n \geqslant 2$, there is an $m \times n$ permutation that admits an infinite sequence of two-dimensional homing operations. (3) If only vertical-first homing operations are allowed, for any integers $m \geqslant 3$ and $n \geqslant 2$, there is an $m \times n$ permutation that admits an infinite sequence of two-dimensional homing operations. (4) The number of $2 \times n$ permutations that admit sequences of $\Omega(2^n)$ vertical-first two-dimensional homing operations is super-exponential in n. (5) No $2 \times n$ permutation admits a sequence of $(2n)!$ or more vertical-first two-dimensional homing operations.

1. Introduction

Let π be an n-element permutation. Let $\pi[i]$ be the element in position i of π. Let $\pi(i)$ be the position of element i in π. Permutation π is *sorted* if $\pi[i]=i$ holds for each $i=1,2,\ldots,n$. Let $home(i)$ be the *one-dimensional homing operation* such that the permutation $\pi' = \pi \circ home(i)$ obtained by applying $home(i)$ on π is as follows: If $\pi[i] < i$, let

$$\pi'[j] = \begin{cases} \pi[j] & \text{if } j < \pi(i) \text{ or } j > i \\ i & \text{if } j = i \\ \pi[j+1] & \text{if } \pi(i) < j < i \end{cases}$$

If $i < \pi[i]$, let

$$\pi'[j] = \begin{cases} \pi[j] & \text{if } j < i \text{ or } j > \pi(i) \\ i & \text{if } j = i \\ \pi[j-1] & \text{if } i < j < \pi(i) \end{cases}$$

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That is, *home*(i) places element i at the i-th position of π and shifts accordingly the elements of π between positions i and $\pi(i)$. For instance, let π_1, π_2, π_3, and π_4 be the following 7-element permutations.

π_1	1	2	3	4	5	6	7
	6	1	3	4	5	2	7

$\xrightarrow{home(6)}$

π_2	1	2	3	4	5	6	7
	1	3	4	5	2	6	7

$\swarrow$ *home*(4)

π_3	1	2	3	4	5	6	7
	1	3	5	4	2	6	7

$\xrightarrow{home(2)}$

π_4	1	2	3	4	5	6	7
	1	2	3	5	4	6	7

We have $\pi_2 = \pi_1 \circ home(6)$, $\pi_3 = \pi_2 \circ home(4)$, and $\pi_4 = \pi_3 \circ home(2)$. We say that permutation π_1 *admits* a sequence of ℓ homing operations $home(i_1), home(i_2), \ldots, home(i_\ell)$ if for each $j = 1, 2, \ldots, \ell$, element i_j is not at the i_j-th position of permutation $\pi_{j+1} = \pi_j \circ home(i_j)$. For instance, in the above example, π_1 admits the sequence $home(6), home(4), home(2)$ of three homing operations. However, π_4 does not admit any sequence of two or more homing operations, since π_4 admits exactly two homing operations $home(4)$ and $home(5)$ and $\pi_4 \circ home(4) = \pi_4 \circ home(5)$ is the sorted permutation, which does not admit any homing operation. Elizalde and Winkler [1] studied one-dimensional homing sort, i.e., sorting via the above "placement-and-shift" one-dimensional homing operations, and obtained the following results: (1) Any n-element permutation can be sorted by at most $n-1$ homing operations. (2) The number of n-element permutations that admit a sequence of $2^{n-1}-1$ homing operations is super-exponential in n. (3) No n-element permutation admits a sequence of 2^{n-1} or more one-dimensional homing operations. Therefore, if one iteratively applies homing operations on an n-element permutation, the permutation has to be sorted in $2^{n-1}-1$ or less iterations.

One-dimensional homing sort can be naturally extended to two dimension. Let π be an $m \times n$ permutation. Let $[i, j]$ be the position at the i-th row and the j-th column. Let $row[i, j] = i$ and $col[i, j] = j$. Let (i, j) be the element whose home position is $[i, j]$. Let $\pi[i, j]$ be the element at position $[i, j]$ of π. Let $\pi(i, j)$ be the position of element (i, j) in π. Permutation π is *sorted* if $\pi[i, j] = (i, j)$ holds for all indices $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, i.e., each element is at its home position. For any element (i, j) of π, we have two kinds of two-dimensional homing operations.

- Let $home_v(i, j)$ be the *vertical-first two-dimensional homing operation*, which (1) vertically places element (i, j) to position $[i, col(\pi(i, j))]$, (2) vertically shifts all the elements between positions $\pi(i, j)$ and $[i, col(\pi(i, j))]$, (3) horizontally places element (i, j) to its home position $[i, j]$, and (4) horizontally shifts all the elements between positions $[i, col(\pi(i, j))]$ and $[i, j]$.
- Let $home_h(i, j)$ be the *horizontal-first two-dimensional homing operation*, which (1) horizontally places element (i, j) to position $[row(\pi(i, j)), j]$, (2) horizontally shifts all the elements between position $\pi(i, j)$ and $[row(\pi(i, j)), j]$, (3) vertically places element (i, j) to its home position $[i, j]$, and (4) vertically shifts all the elements between positions $[row(\pi(i, j)), j]$ and $[i, j]$.

For instance, let π_1, π_2, and π_3 be as follows.

π_1	1	2	3
1	(1,2)	(2,3)	(1,1)
2	(1,3)	(2,2)	(2,1)

$\xrightarrow{home_v(1,3)}$

π_2	1	2	3
1	(2,3)	(1,1)	(1,3)
2	(1,2)	(2,2)	(2,1)

π_1	1	2	3
1	(1,2)	(2,3)	(1,1)
2	(1,3)	(2,2)	(2,1)

$\xrightarrow{home_h(1,3)}$

π_3	1	2	3
1	(1,2)	(2,3)	(1,3)
2	(2,2)	(2,1)	(1,1)

We have $\pi_2 = \pi_1 \circ home_v(1,3)$ and $\pi_3 = \pi_1 \circ home_h(1,3)$. Permutation π_1 *admits* a sequence of ℓ homing operations $home_{x_1}(i_1, j_1), home_{x_2}(i_2, j_2), \ldots, home_{x_\ell}(i_\ell, j_\ell)$ if for each $k = 1, 2, \ldots, \ell$, element (i_k, j_k) is not at position $[i_k, j_k]$ of permutation $\pi_{k+1} = \pi_k \circ home_{x_k}(i_k, j_k)$. In the present paper, we make an initial attempt in studying sorting via two-dimensional homing operations and obtain the following observations.

Theorem 1.1.

1. *Any $m \times n$ permutation can be sorted by at most $mn - 1$ two-dimensional homing operations.*
2. *If both vertical-first and horizontal-first homing operations are allowed, for any integers $m \geqslant 2$ and $n \geqslant 2$, there is an $m \times n$ permutation that admits an infinite sequence of two-dimensional homing operations.*
3. *If only vertical-first homing operations are allowed, for any integers $m \geqslant 3$ and $n \geqslant 2$, there is an $m \times n$ permutation that admits an infinite sequence of two-dimensional homing operations.*
4. *The number of $2 \times n$ permutations that admit a sequence of $\Omega(2^n)$ vertical-first two-dimensional homing operations is super-exponential in n.*
5. *No $2 \times n$ permutation admits a sequence of $(2n)!$ or more vertical-first two-dimensional homing operations.*

For related work of one-dimensional homing sort, see Elizalde and Winkler [1] and the references therein. In particular, a similar Topswops algorithm of Conway was discussed by Gardner [2, p. 76].

The rest of the paper is organized as follows. Section 2 gives the preliminaries. Section 3 proves the theorem of the paper. Section 4 concludes the paper with a couple of open questions.

2. Preliminaries

Let $count_h(\pi, i)$ be the number of elements in the first i rows of π whose home positions are also in the first i rows of π. Let $count_v(\pi, j)$ be the number of elements in the first j columns of π whose home positions are also in the first j columns of π.

Lemma 2.1. *If π and π' are $m \times n$ permutations such that $\pi = \pi' \circ home_v(i, j)$ holds for some element (i, j), then $count_v(\pi', c) \leqslant count_v(\pi, c)$ holds for each $c = 1, 2, \ldots, n$ and $count_h(\pi', r) \leqslant count_h(\pi, r)$ holds for each $r = 1, 2, \ldots, m$.*

Proof. We prove the inequality for $count_v$. The inequality for $count_h$ can be proved similarly. Since the statement holds trivially when $c = n$ or $count_v(\pi', c) = count_v(\pi, c)$, the proof focuses on the cases with $1 \leqslant c < n$ and $count_v(\pi', c) \neq count_v(\pi, c)$. Suppose that $\pi = \pi' \circ home_v(i, j)$. Since $\pi[i, j] = (i, j)$, we have $j = col(\pi(i, j))$. Let $j' = col(\pi'(i, j))$. Let $i' = row(\pi'(i, j))$. By $count_v(\pi, c) \neq count_v(\pi', c)$, operation $home_v(i, j)$ moves some elements across the boundary between columns c and $c + 1$, implying that either $j \leqslant c < j'$ or $j' \leqslant c < j$ holds.

Case 1. $j \leqslant c < j'$. Operation $home_v(i, j)$ moves element (i, j) into the first c columns and shifts element $\pi[i, c]$ out of the first c columns. By $j \leqslant c$, element (i, j) is in the first c columns of π. If element (i, c) is also in the first c columns of π, then $count_v(\pi', c) = count_v(\pi, c)$; otherwise, $count_v(\pi', c) < count_v(\pi, c)$.

Case 2. $j' \leqslant c < j$. Operation $home_v(i, j)$ moves element (i, j) out of the first c columns and shifts element $\pi[i, c + 1]$ into the first c columns. By $j > c$, element (i, j) is not in the first c columns of π. If element $(i, c + 1)$ is not in the first c columns of π, then $count_v(\pi', c) = count_v(\pi, c)$; otherwise, $count_v(\pi', c) < count_v(\pi, c)$. □

3. Our proof

This section proves Theorem 1.1.

Proof. To see Theorem 1.1(1), one can verify that any $m \times n$ permutation π can be sorted by the following mn iterations: For each $\ell = 1, 2, \ldots, mn$, if $\pi(i, j) \neq [i, j]$ holds for the numbers i and j with $\ell = i + (j - 1) \cdot m$, then we replace π by $\pi \circ home_v(i, j)$.

To see Theorem 1.1(2), let $\pi_1, \pi_2, \pi_3, \pi_4$ be as follows.

π_1	1	2
1	$(2,1)$	$(1,2)$
2	$(2,2)$	$(1,1)$

$\rightarrow$

π_2	1	2
1	$(1,1)$	$(2,1)$
2	$(2,2)$	$(1,2)$

$\downarrow$

π_3	1	2
1	$(2,2)$	$(1,1)$
2	$(2,1)$	$(1,2)$

$\leftarrow$

π_4	1	2
1	$(2,1)$	$(1,1)$
2	$(1,2)$	$(2,2)$

$\uparrow$

One can verify $\pi_2 = \pi_1 \circ home_v(1, 1)$, $\pi_3 = \pi_2 \circ home_h(2, 1)$, $\pi_4 = \pi_3 \circ home_v(2, 2)$, and $\pi_1 = \pi_4 \circ home_h(1, 2)$.

To see Theorem 1.1(3), let $\pi_1, \pi_2, \dots, \pi_8$ be as follows.

π_1	1	2
1	(2, 2)	(1, 2)
2	(2, 1)	(3, 1)
3	(1, 1)	(3, 2)

→

π_2	1	2
1	(2, 2)	(1, 2)
2	(2, 1)	(3, 2)
3	(3, 1)	(1, 1)

→

π_3	1	2
1	(2, 1)	(1, 2)
2	(3, 2)	(2, 2)
3	(3, 1)	(1, 1)

→

π_4	1	2
1	(1, 1)	(2, 1)
2	(3, 2)	(1, 2)
3	(3, 1)	(2, 2)

↓

π_5	1	2
1	(1, 1)	(2, 1)
2	(3, 2)	(2, 2)
3	(3, 1)	(1, 2)

→

π_6	1	2
1	(1, 1)	(2, 1)
2	(3, 1)	(2, 2)
3	(1, 2)	(3, 2)

→

π_7	1	2
1	(1, 1)	(2, 2)
2	(2, 1)	(3, 1)
3	(1, 2)	(3, 2)

→

π_8	1	2
1	(2, 2)	(1, 2)
2	(1, 1)	(3, 1)
3	(2, 1)	(3, 2)

↑ (π_1)

We have $\pi_2 = \pi_1 \circ home_v(3,1)$, $\pi_3 = \pi_2 \circ home_v(2,2)$, $\pi_4 = \pi_3 \circ home_v(1,1)$, $\pi_5 = \pi_4 \circ home_v(2,2)$, $\pi_6 = \pi_5 \circ home_v(3,2)$, $\pi_7 = \pi_6 \circ home_v(2,1)$, $\pi_8 = \pi_7 \circ home_v(1,2)$, and $\pi_1 = \pi_8 \circ home_v(2,1)$. Hence we have an infinite sequence of two-dimensional homing operations for some 3×2 permutation, which can be embedded into an $m \times n$ permutation for any integers $m \geqslant 3$ and $n \geqslant 2$. Theorem 1.1(3) is proved.

Theorem 1.1(4) follows from result (3) of Elizalde and Winkler [1] by considering those $2 \times n$ permutations π such that $row(\pi(i,j)) = i$ holds for each $i = 1, 2$ and $j = 1, 2, \dots, n$.

The rest of the section lets $home(i,j) = home_v(i,j)$ and proves Theorem 1.1(5). Assume for contradiction that some $2 \times n$ permutation admits a sequence of more than $(2n)!$ vertical-first two-dimensional homing operations. Since there are exactly $(2n)!$ distinct $2 \times n$ permutations, there are ℓ distinct $2 \times n$ permutations $\pi_1, \pi_2, \dots, \pi_\ell = \pi_0$ for some integer ℓ with $2 \leqslant \ell \leqslant (2n)!$ such that

$$\pi_{k+1} = \pi_k \circ home(i_k, j_k)$$

holds for each $k = 0, 1, \dots, \ell - 1$. There is an integer k with $0 \leqslant k \leqslant \ell - 1$ such that $col(\pi_k(i_k, j_k)) \neq j_k$, since otherwise this homing sequence would yield a 1-dimensional homing cycle, contradicting with result (3) of Elizalde and Winkler [1]. Without loss of generality, we assume that an element (i_k, j_k) with $0 \leqslant k \leqslant \ell - 1$ is homed to the left in the above homing cycle. That is,

$$L = \big\{k \bigm| 0 \leqslant k \leqslant \ell - 1 \text{ and } col\big(\pi_{k+1}(i_k, j_k)\big) < col\big(\pi_k(i_k, j_k)\big)\big\}$$

is non-empty. Without loss of generality, we may assume $0 \in L$, $i_0 = 1$, and $j_0 \leqslant j_k$ holds for each $k \in L$. Hence we have $1 \leqslant j_0 \leqslant n - 1$. By Lemma 2.1, equation

$$count_h(\pi_{k+1}, 1) = count_h(\pi_k, 1) \tag{1}$$

holds for each $k = 0, 1, \dots, \ell - 1$. Since $\pi_0 = \pi_\ell$, by Lemma 2.1, equation

$$count_v(\pi_{k+1}, c) = count_v(\pi_k, c) \tag{2}$$

holds for each $k = 0, 1, \dots, \ell - 1$ and each $c = 1, 2, \dots, n$. We have $\pi_1[1, j_0] = (1, j_0)$, so π_1 is as follows.

π_1	$\cdots$	j_0	$\cdots$
1	$\cdots$	$(1, j_0)$	$\cdots$
2	$\cdots$		$\cdots$

By $0 \in L$, element $(1, j_0)$ is shifted from the j_0-th column to the $(j_0 + 1)$-st column at some point. The only way to shift element $(1, j_0)$ out of the first j_0 columns is by operation $home(2, j_0)$ which shifts element $(1, j_0)$ from position $[2, j_0]$ to position $[2, j_0 + 1]$.

Let k_0 be the smallest integer such that π_{k_0} is as follows.

π_{k_0}	$\cdots$	j_0	$j_0 + 1$	$\cdots$
1	$\cdots$			$\cdots$
2	$\cdots$	$(2, j_0)$	$(1, j_0)$	$\cdots$

Let k^* be the largest integer with $k_0 \leqslant k^*$ such that element $(1, j_0)$ is in the second row of permutations $\pi_{k_0}, \pi_{k_0+1}, \dots, \pi_{k^*}$. By Eq. (1) and the definition of j_0, the only way for element $(1, j_0)$ to leave the second row is through operation $home(1, j_0)$, implying $(i_{k^*}, j_{k^*}) = (1, j_0)$. Element $(2, j_0)$ is not at its home position in permutation π_{k^*}, since otherwise elements $(1, j_0)$ and $(2, j_0)$ would never get out of the first j_0 columns for the rest of the homing cycle, a contradiction. Let k_0^* be the largest integer with $k_0 \leqslant k_0^* < k^*$ such that element $(2, j_0)$ is at its home position $[2, j_0]$ in permutations $\pi_{k_0}, \pi_{k_0+1}, \ldots, \pi_{k_0^*}$. By the definitions of L and j_0, operation $home(i_{k_0^*}, j_{k_0^*})$ shifts element $(2, j_0)$ from position $[2, j_0]$ to position $[1, j_0]$ or $[2, j_0 - 1]$. We have $i_{k_0^*} = 2$, $j_{k_0^*} \geqslant j_0 + 1$, and $col(\pi_{k_0^*}(i_{k_0^*}, j_{k_0^*})) \leqslant j_0$. Therefore, element $(1, j_0)$ is not at position $[1, j_0 + 1]$ in permutation $\pi_{k_0^*}$, since otherwise we would have $count_v(\pi_{k_0^*+1}, j_0) = count_v(\pi_{k_0^*}, j_0) + 1$, contradicting with Eq. (2).

Let k_1 be the smallest integer with $k_0 < k_1 \leqslant k_0^*$ such that element $(1, j_0)$ is not at position $[2, j_0+1]$ in permutation π_{k_1}. By $k_0 < k_1 \leqslant k_0^* < k^*$, element $(2, j_0)$ is at its home position in π_{k_1} and element $(1, j_0)$ is in the second row of π_{k_1}. Thus, operation $home(i_{k_1-1}, j_{k_1-1})$ shifts element $(1, j_0)$ to position $[2, j_0 + 2]$. Hence we know $j_0 \leqslant n - 2$ and $(i_{k_1-1}, j_{k_1-1}) = (2, j_0 + 1)$. Permutation π_{k_1} is as follows.

π_{k_1}	$\cdots$	j_0	j_0+1	j_0+2	$\cdots$
1	$\cdots$				$\cdots$
2	$\cdots$	$(2, j_0)$	$(2, j_0+1)$	$(1, j_0)$	$\cdots$

Let k_1^* be the largest integer with $k_1 \leqslant k_1^* \leqslant k_0^*$ such that element $(2, j_0 + 1)$ is at its home position in permutations $\pi_{k_1}, \pi_{k_1+1}, \ldots, \pi_{k_1^*}$. Since operation $home(i_{k_1^*}, j_{k_1^*})$ shifts element $(2, j_0 + 1)$ out of position $[2, j_0 + 1]$, it follows from the definitions of j_0 and L that $i_{k_1^*} = 2$, $j_{k_1^*} \geqslant j_0 + 2$, and $col(\pi_{k_1^*}(i_{k_1^*}, j_{k_1^*})) \leqslant j_0 + 1$. Thus, $\pi_{k_1^*}(1, j_0) \neq [1, j_0 + 2]$, since otherwise we would have $count_v(\pi_{k_1^*+1}, j_0 + 1) = count_v(\pi_{k_1^*}, j_0 + 1) + 1$, contradicting with Eq. (2). It follows that there is a smallest integer k_2 with $k_1 < k_2 \leqslant k_1^*$ such that element $(1, j_0)$ is not at position $[2, j_0 + 2]$ in permutation π_{k_2}. By $k_0 < k_1 < k_2 \leqslant k_1^* \leqslant k_0^* \leqslant k^*$, elements $(2, j_0)$ and $(2, j_0 + 1)$ are at their home positions in π_{k_2} and element $(1, j_0)$ is in the second row of π_{k_2}. Therefore, operation $home(i_{k_2-1}, j_{k_2-1})$ shifts element $(1, j_0)$ from position $[2, j_0 + 2]$ to position $[2, j_0 + 3]$. Hence we know $j_0 \leqslant n - 3$ and $(i_{k_2-1}, j_{k_2-1}) = (2, j_0 + 2)$. Permutation π_{k_2} is as follows.

π_{k_2}	$\cdots$	j_0	j_0+1	j_0+2	j_0+3	$\cdots$
1	$\cdots$					$\cdots$
2	$\cdots$	$(2, j_0)$	$(2, j_0+1)$	$(2, j_0+2)$	$(1, j_0)$	$\cdots$

By continuing the above argument, there is a smallest integer k with $k_0 < k < k^*$ such that π_k is as below and $(i_k, j_k) = (2, n)$ and $col(\pi_k(2, n)) \leqslant n - 1$ hold.

π_k	$\cdots$	j_0	j_0+1	j_0+2	$\cdots$	$n-1$	n
1	$\cdots$				$\cdots$		
2	$\cdots$	$(2, j_0)$	$(2, j_0+1)$	$(2, j_0+2)$	$\cdots$	$(2, n-1)$	$(1, j_0)$

We have $count_v(\pi_{k+1}, n - 1) = count_v(\pi_k, n - 1) + 1$, contradicting with Eq. (2). Theorem 1.1(5) is proved. □

4. Concluding remarks

It would be of interest to see bounds tighter than Theorems 1.1(4) and 1.1(5). Following Elizalde and Winkler [1], we leave open the exact number of worst-case $2 \times n$ permutations.

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