



# The longest almost-increasing subsequence

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## ABSTRACT

Given a sequence of  $n$  elements, we introduce the notion of an almost-increasing subsequence as the longest subsequence that can be converted to an increasing subsequence by possibly adding a value, that is at most a fixed constant, to each of the elements. We show how to optimally construct such subsequence in  $O(n \log k)$  time, where  $k$  is the length of the output subsequence.

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## 1. Introduction

The longest increasing subsequence (LIS) is a subsequence of maximum length where every element is greater than the previous element. The longest increasing subsequence problem refers to either producing the subsequence or just finding its length. The problem was first tackled by Robinson [16] seventy years ago. The classical dynamic-programming algorithm for the problem, which appears in many algorithmic textbooks [9], is due to Schensted [17]. This algorithm runs in  $O(n \log k)$ , where  $n$  is the length of the input sequence, and  $k$  is the length of the longest increasing subsequence. Knuth [14] gave generalizations to the problem with relations to Young tableaux. Fredman [12] showed that  $O(n \log n)$  comparisons are both necessary and sufficient, to find the length or produce the subsequence, in the comparison-tree model. The same lower bound was also proven for the algebraic decision-tree model [15]. If the input sequence is a permutation of the integers 1 to  $n$ , algorithms were introduced to construct the longest increasing subsequence in  $O(n \log \log n)$  time [8,13].

The problem is important in practice. Several other problems involve a LIS construction (see, for example, [5]).

It has lately gained even more practical importance as it is used in the MUMmer system [10] for aligning genomes.

A related problem is the longest common subsequence (LCS) problem, which considers two sequences and locates a series of entries that appear in the same order in both sequences. Note that we can apply the LCS algorithms to a sequence and its sorted outcome to get a longest increasing subsequence.

Several variants of the LIS problem have been introduced. The longest increasing subsequence of a circular list (LICS) assumes the input sequence to be circular. A randomized algorithm for the LICS that runs in expected  $O(n^{3/2} \log n)$  time is given in [3]. The best worst-case bound known is  $O(n^2)$ , and can be achieved using techniques from [4]. Another variant is to find the longest increasing subsequences that lie in all the sliding windows with a specified width. An algorithm that runs in time proportional to the size of the output subsequences plus an additive bound for constructing one LIS is given in [4]. A generalization of the LIS problem is discussed in [2], where a fixed set of permutations is given and the task is to compute, for a given input sequence, the longest subsequence that is order isomorphic to one of the given permutations. The LIS problem applies when such permutations are the identity permutations.

The combinatorics of the problem are of no less importance. Starting with the work of Erdős and Szekeres, the

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length of a LIS in a random permutation was investigated. The complete limiting distribution of the length of the LIS of a permutation of length  $n$  chosen uniformly at random is given in [6]. The expected length of a LIS is shown to be close to  $2\sqrt{n}$ .

Suppose one is considering the process of monitoring the performance of an activity. We say that the activity is well performing once it is well performing in comparison with a large number of accredited historical snapshots where it was as well performing when deploying the same criteria. Picking the largest number of points when the activity is strictly performing better among such previously selected points is too restricted and unfair. The notion needs to be relaxed to reflect a good progress without necessarily being the best selected so far. On another front, there are applications where the data items have a small amount of noise. In accordance, a relaxed version of the LIS problem is needed.

In this paper, we introduce a variant of the LIS problem that we call the longest almost-increasing subsequence (LaIS) problem. We allow a drop of at most a constant value from the maximum element that appeared so far. Given a sequence  $\langle x_1, x_2, \dots, x_n \rangle$  and a constant  $c > 0$ , our goal is to construct a longest subsequence  $\langle y_1, y_2, \dots, y_k \rangle$  such that  $\forall_i y_i > \max_{j=1}^{i-1} y_j - c$ . We give an asymptotically optimal algorithm that runs in  $O(n \log k)$  time. The main idea is to apply the dynamic programming paradigm to a search-utilizing pointer-based data structure and not to an array.

## 2. Recursive formulation

Let  $\text{LaIS}(h, i)$ ,  $h \leq i$ , denote a longest almost-increasing subsequence among the elements of  $\langle x_1, \dots, x_i \rangle$ , such that a largest element is  $x_h$  and  $h$  is minimal. We show that these two parameters fully characterize any LaIS, and recursively express LaIS in terms of solutions to smaller problems.

The key observation is that any  $\text{LaIS}(h, i)$ ,  $h < i$ , can be split into two independent (except for the value of  $h$ ) subsequences, following the relation:

$$\text{LaIS}(h, i) = \text{LaIS}(h, h) \cdot T(h, i),$$

where “ $\cdot$ ” is the concatenation operation, and  $T(h, i)$  is the subsequence including every element  $x_j$  among  $\langle x_{h+1}, \dots, x_i \rangle$  satisfying  $x_h - c < x_j \leq x_h$ .

The second observation is that  $\text{LaIS}(h, h)$  can be expressed as:

$$\text{LaIS}(h, h) = \text{LaIS}(i', h-1) \cdot \langle x_h \rangle,$$

where  $\text{length}(\text{LaIS}(i', h-1))$  has the maximum value among  $i' < h$  satisfying  $x_{i'} < x_h$ . Note that  $i'$  is not necessarily unique.

## 3. The basic algorithm

The algorithm proceeds in  $n$  iterations. After the  $i$ -th iteration, the algorithm maintains for each element  $x_j$ , among the first  $i$  elements, a longest almost-increasing

subsequence whose largest element is  $x_j$  and  $j$  is minimal. For each such subsequence, it is enough to keep track of its length  $l_j$  and the minimal index  $p_j$  of a largest element among the elements preceding  $x_j$ . During the  $i$ -th iteration, two tasks are performed: The **first task** is to find a longest almost-increasing subsequence whose largest element is  $x_i$ . This subsequence is constructed by appending  $x_i$  to the longest subsequence found so far whose largest element is smaller than  $x_i$ . More formally, we look for an index  $i' < i$  such that  $l_{i'} \geq l_j$  among all indexes  $j < i$  having  $x_j < x_i$ . The length  $l_i$  is then set to  $l_{i'} + 1$ , and the index  $p_i$  is set to  $i'$ . The **second task** is to append  $x_i$  to every subsequence found so far whose largest element is larger or equal to  $x_i$  and smaller than  $x_i + c$ . More formally, for all  $j < i$ , set  $l_j$  to  $l_j + 1$  if  $x_i \leq x_j < x_i + c$ . After the  $n$ -th iteration, the length of the LaIS is the maximum length  $l_m$  among the  $l_i$ 's stored by the algorithm. To construct a LaIS, we make use of the  $p_i$ 's to produce the subsequence in reverse order. Using the element  $x_m$  corresponding to the maximum length  $l_m$ , scan every element  $x_i$ , from  $i = n$  to  $m + 1$ , and output the elements satisfying  $x_m - c < x_i \leq x_m$  followed by  $x_m$ . Let  $x_t$  be the last element of the subsequence output in the reverse order, scan every element  $x_i$  from  $i = t - 1$  to  $p_t + 1$ , and output the elements satisfying  $x_{p_t} - c < x_i \leq x_{p_t}$  followed by  $x_{p_t}$ . The previous step is repeated until we get a value of  $p_t$  that indicates the first element of the subsequence (when, for example,  $p_t = t$ ).

A straightforward implementation of the previous algorithm would use two linked lists (or two arrays) each of size  $n$ ; one for the  $l_i$ 's and another for the  $p_i$ 's. This implementation runs in  $O(n^2)$  time.

### Example

Assume  $c = 2$ , and consider the following sequence indexed from 1 to 12:  $\langle 7, 15, 2, 14, 14, 6, 8, 11, 17, 15, 14, 13 \rangle$ . We show in Table 1 the two arrays: one holding the lengths  $l_i$ 's (first row), and another holding the indexes  $p_i$ 's (second row), after each of the 12 iterations performed by the algorithm.

It follows that the LaIS is of length 6. There are five such subsequences:  $\langle 7, 15, 14, 14, 15, 14 \rangle$ ,  $\langle 7, 6, 8, 11, 15, 14 \rangle$ ,  $\langle 2, 6, 8, 11, 15, 14 \rangle$ ,  $\langle 7, 6, 8, 11, 14, 13 \rangle$  and  $\langle 2, 6, 8, 11, 14, 13 \rangle$ . Typically, the algorithm stores one  $p_i$  and reports one LaIS.

## 4. The improved algorithm

Instead of storing one  $l_i$  corresponding to each  $x_i$ , we store one element for every length value. Namely, after the  $i$ -th iteration, we store  $z_l \leftarrow x_h$  corresponding to length  $l$ , where  $\text{length}(\text{LaIS}(h, i)) = l$  and  $x_h \leq x_{h'}$  for all  $h'$  satisfying  $\text{length}(\text{LaIS}(h', i)) = l' \geq l$ . It follows that  $z_j \leq z_{j'}$  for  $j < j'$ . To show that these elements are enough to construct a LaIS, consider two elements  $x_h > x_{h'}$  where  $\text{length}(\text{LaIS}(h', i)) \geq \text{length}(\text{LaIS}(h, i))$ . Given any almost-increasing subsequence having  $\text{LaIS}(h, i)$  as a prefix subsequence, we can replace  $\text{LaIS}(h, i)$  with  $\text{LaIS}(h', i)$  and get another almost-increasing subsequence with at least the same length.



### Example

Back to the example of Section 3, where  $c = 2$  and the input sequence is  $\langle 7, 15, 2, 14, 14, 6, 8, 11, 17, 15, 14, 13 \rangle$ . We show in Table 2 the two arrays: one holding the  $z_i$ 's (part a), and another holding the indexes  $p_i$ 's (part b), after each of the 12 iterations performed by the algorithm.

The LaS reported by this version of the algorithm will be  $\langle 2, 6, 8, 11, 14, 13 \rangle$ .

### 5. The data structure

In order to achieve the claimed  $O(n \log k)$  bound, the inner loop must be executed in  $O(\log k)$  time. To efficiently implement a length-shift, we store the  $z_j$ 's as an ordered linked list, where a node holding  $z_j$  points to a node holding its successor element  $z_{j+1}$ . During each iteration, a search is performed for the predecessor of  $x_i$ . A node that contains  $x_i$  is then inserted after this position.

To perform a length-shift, the corresponding range of nodes is determined and the successor of the last node in the range is deleted (if such a node exists). To construct the subsequence, we still maintain an array for the  $p_i$ 's. This array is modified once per iteration and later used to produce the output.

In summary, at each iteration the algorithm performs: two predecessor searches, one successor finding, an insertion and a possible deletion.

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#### Algorithm 1 The pseudo-code for the LaS algorithm.

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1: for  $i = 1$  to  $n$  do
2:    $v \leftarrow \text{new\_node}()$ 
3:    $v.\text{value} \leftarrow x_i$ 
4:    $v.\text{index} \leftarrow i$ 
5:    $\text{pred} \leftarrow \text{predecessor}(x_i)$ 
6:   if  $(\text{pred} \neq \text{null})$  then
7:      $p_i \leftarrow \text{pred.index}$ 
8:   else
9:      $p_i \leftarrow i$ 
10:  end if
11:  insert( $v$ )
12:   $s \leftarrow \text{successor}(\text{predecessor}(x_i + c))$ 
13:  if  $(s \neq \text{null})$  then
14:    delete( $s$ )
15:  end if
16: end for
17:  $m \leftarrow \text{tail\_node}().\text{index}$ 
18: for  $i = n$  down to  $m + 1$  do
19:   if  $(x_m - c < x_i \leq x_m)$  then
20:     print  $x_i$ 
21:   end if
22: end for
23: print  $x_m$ 
24:  $t \leftarrow m$ 
25: while  $(p_t \neq t)$  do
26:   for  $i = t - 1$  down to  $p_t + 1$  do
27:     if  $(x_{p_t} - c < x_i \leq x_{p_t})$  then
28:       print  $x_i$ 
29:     end if
30:   end for
31:   print  $x_{p_t}$ 
32:    $t \leftarrow p_t$ 
33: end while

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Each of these operations can be executed in  $O(\log k)$  time when building a balanced search structure over the

nodes of the linked list. We may use, for example, an AVL tree [1] to achieve  $O(\log k)$  cost per operation, or a splay tree [18] for an amortized  $O(\log k)$  cost per operation. For practical purposes, it would be better to keep a linked list of pointers to find the successor of a given node in constant time, accounting for at least three pointers per node in case using the aforementioned structures. However, from the practical point of view, the best search structure for this application is the jumplist [11]. Using a jumplist: a predecessor search, an insertion and a deletion require  $O(\log k)$  amortized time ( $O(\log k)$  expected time [7]), finding the successor requires constant time, and we only use and maintain two pointers per node.

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### References

- [1] G. Adelson-Velskii, E. Landis, On an information organization algorithm, Doklady Akademii Nauk SSSR 146 (1962) 263–266.
- [2] M. Albert, R. Aldred, M. Atkinson, H. Ditmarsch, B. Handley, C. Handley, J. Opatrny, Longest subsequences in permutations, Australian Journal of Combinatorics 28 (2003) 225–238.
- [3] M. Albert, M. Atkinson, D. Nussbaum, J. Sack, N. Santoro, On the longest increasing subsequence of a circular list, Information Processing Letters 101 (2007) 55–59.
- [4] M. Albert, A. Golinski, A. Hamel, A. Lopez-Ortiz, S. Rao, M. Safari, Longest increasing subsequences in sliding windows, Theoretical Computer Science 321 (2004) 405–414.
- [5] A. Apostolico, M. Atallah, S. Hambrusch, New clique and independent set algorithms for circle graphs, Discrete Applied Mathematics 36 (1992) 1–24.
- [6] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, Journal of the American Mathematical Society 12 (1999) 1119–1178.
- [7] H. Brönnimann, F. Cazals, M. Durand, Randomized jumplists: A jump-and-walk dictionary data structure, in: 20th Symposium on Theoretical Aspects of Computer Science, in: LNCS, vol. 2607, 2003, pp. 283–294.
- [8] M. Chang, F. Wang, Efficient algorithms for the maximum weight clique and maximum weight independent set problems on permutation graphs, Information Processing Letters 76 (2000) 7–11.
- [9] T. Cormen, C. Leiserson, R. Rivest, C. Stein, Introduction to Algorithms, 2nd ed., The MIT Press, Cambridge, 2001.
- [10] A. Delcher, S. Kasif, R. Fleischmann, J. Paterson, O. White, S. Salzberg, Alignment of whole genomes, Nucleic Acids Research 27 (1999) 2369–2376.
- [11] A. Elmasry, Deterministic jumplists, Nordic Journal of Computing 12 (1) (2005) 27–39.
- [12] M. Fredman, On computing the length of longest increasing subsequence, Discrete Mathematics 11 (1975) 29–35.
- [13] J. Hunt, T. Szymanski, A fast algorithm for computing longest common subsequences, Communications of the ACM 20 (1977) 350–353.
- [14] D. Knuth, Permutations, matrices, and generalized Young tableaux, Pacific Journal of Mathematics 34 (1970) 709–727.
- [15] P. Ramanan, Tight  $\Omega(n \log n)$  lower bound for finding a longest increasing subsequence, International Journal of Computer Mathematics 65 (3–4) (1997) 161–164.
- [16] G. Robinson, On representations of the symmetric group, American Journal of Mathematics 60 (1938) 745–760.
- [17] C. Schensted, Longest increasing and decreasing subsequences, Canadian Journal of Mathematics 13 (1961) 179–191.
- [18] D. Sleator, R. Tarjan, Self-adjusting binary search trees, Journal of the ACM 32 (3) (1985) 652–686.