# Displacements of Weighted Graphs *

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**Abstract**

*In this paper we investigate the relationship between the status and the displacement of a connected weighted graph. We also obtain a product result for the displacements.*

## 1 Introduction

All the graphs considered in this paper are finite and simple. We introduce two parameters of graphs: displacement and status.

We first give the definition of the displacement of a graph. Let $G$ be a connected graph and let $\phi$ be a permutation of $V(G)$ (i.e., $\phi$ is a bijection from $V(G)$ to $V(G)$). The *displacement* $D_G(\phi)$ of $\phi$ is defined by

$$D_G(\phi) = \sum_{x \in V(G)} d_G(x, \phi(x)).$$

The *displacement* $D(G)$ of $G$ is defined by

$$D(G) = max\ D_G(\phi),$$

where the maximum is taken over all permutations $\phi$ of $V(G)$.

E.T.H. Wang proposed a problem [5] which is equivalent to determining the displacements of all permutations of the path $P_n$ (a path on $n$ vertices). He and other solvers obtained that all the possible values are $0, 2, 4, \cdots, \lfloor \frac{n^2}{2} \rfloor$.

Next we give the definition of the status of a graph. Let $G$ be a connected graph. For a vertex $x$ in $G$, the *status* $s_G(x)$ of $x$ is defined by

$$s_G(x) = \sum_{y \in V(G)} d_G(x, y).$$

The *status* $s(G)$ of the graph $G$ is defined by

$$s(G) = \min_{x \in V(G)} s_G(x).$$

The *median* of $G$ is the set of vertices in $G$ with status equal to $s(G)$ (i.e, the set of vertices with minimum status).

For the results about statuses one may refer to [1]. The following characterization of the median of a tree follows from the results of A. Kang and D. Ault [2], and was mentioned by O. Kariv and S.L. Hakimi in [4] for trees with weights on the vertices and edges.

**Proposition 1.1** [3] *Let $T$ be a tree and $v$ be a vertex in $T$. Then $v$ is in the median of $T$ if and only if $|V(T')| \le \frac{1}{2}|V(T)|$ for every component $T'$ of $T - v$.* □

In the following the notions of displacement and status are extended to the graphs with weights on the edges.

If $G$ is a graph and there exists a weight function $w : E(G) \to R^+$, then $(G, w)$ is called a *weighted graph*.

Let $(G, w)$ be a connected weighted graph. For a path $P$ in $(G, w)$ the *weight* $w_G(P)$ of $P$ is defined by

$$w_G(P) = \sum_{e \in E(P)} w(e).$$

For two vertices $x, y$ in $(G, w)$, the *weight distance* $d_{G,w}(x, y)$ between $x$ and $y$ is defined by

$$d_{G,w}(x, y) = min\ w_G(P),$$

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where the minimum is taken over all paths $P$ which join $x$ and $y$. Note that if $w(e) = 1$ for every edge $e$ in $(G, w)$, then the weight distance is the distance in usual sense, i.e. $d_{G,w}(x, y) = d_G(x, y)$.

It is easy to see that for a connected weighted graph $(G, w)$, we have $d_{G,w}(x, z) + d_{G,w}(z, y) \geq d_{G,w}(x, y)$ where $x, y, z \in V(G)$, and if $v$ is a cut vertex of $G$ and $x, y$ are vertices in different components of $G - v$, then $d_{G,w}(x, v) + d_{G,w}(v, y) = d_{G,w}(x, y)$.

Let $(G, w)$ be a connected weighted graph and let $\phi$ be a permutation of $V(G)$. The *displacement* $D_{G,w}(\phi)$ of $\phi$ is defined by

$$D_{G,w}(\phi) = \sum_{x \in V(G)} d_{G,w}(x, \phi(x)).$$

The *displacement* $D(G, w)$ of $(G, w)$ is defined by

$$D(G, w) = max\ D_{G,w}(\phi),$$

where the maximum is taken over all permutations $\phi$ of $V(G)$.

For a vertex $x$ in a connected weighted graph $(G, w)$, the *status* $s_{G,w}(x)$ of $x$ is

$$s_{G,w}(x) = \sum_{y \in V(G)} d_{G,w}(y, x)\,.$$

The *status* $s(G, w)$ of the graph $(G, w)$ is

$$s(G, w) = min_{x \in V(G)} s_{G,w}(x).$$

The *median* of $(G, w)$ is the set of vertices in $(G, w)$ with status equal to $s(G, w)$. If $w(e) = 1$ for every edge $e$ in $(G, w)$, then $s_{G,w}(x) = s_G(x)$ for all $x \in V(G)$, $s(G, w) = s(G)$, and the median of $(G, w)$ is the same as the median of $G$.

**Theorem 1.2** *Suppose that $(G, w)$ is a connected weighted graph which has a cut vertex $v$ such that $|V(G')| \leq \frac{1}{2}|V(G)|$ for every component $G'$ of $G - v$. Then $v$ is in the median of $(G, w)$.*

**Proof.** Suppose that $x$ is an arbitrary vertex of $G$ other than $v$, and $x \in V(G')$ for some component $G'$ of $G - v$. Then

$$\begin{aligned} & s_{G,w}(x) - s_{G,w}(v) \\ = & \sum_{y \in V(G)} d_{G,w}(y, x) - \sum_{y \in V(G)} d_{G,w}(y, v) \\ = & \sum_{y \in V(G)} (d_{G,w}(y, x) - d_{G,w}(y, v)) \\ = & \sum_{y \in V(G')} (d_{G,w}(y, x) - d_{G,w}(y, v)) \\ & + \sum_{y \in V(G) - V(G')} (d_{G,w}(y, x) - d_{G,w}(y, v)) \\ \geq & \sum_{y \in V(G')} -d_{G,w}(x, v) + \\ & \sum_{y \in V(G) - V(G')} d_{G,w}(v, x) \\ = & (|V(G)| - 2|V(G')|)\ d_{G,w}(v, x) \\ \geq & \ 0. \end{aligned}$$

Thus $s_{G,w}(x) \geq s_{G,w}(v)$ for all vertices $x \neq v$. Hence $v$ is in the median of $(G, w)$. □

In this paper, we investigate the relationship between the status and the displacement of a connected weighted graph and also the displacement about the product of two weighted graphs.

## 2 Relationship between Displacement and Status

In this section, we investigate the relationship between the displacement and the status of a connected weighted graph. The displacement of a graph being the maximum of the displacements of all permutations of the vertices in the graph and the status being the minimum of the statuses of all vertices, the following is an min-max inequality for these two variants.

**Theorem 2.1** *Suppose that $(G, w)$ is a connected weighted graph. Then*

$$D(G, w) \leq 2s(G, w).$$

**Proof.** Let $\phi$ be an arbitrary permutation of $V(G)$, and $y$ be an arbitrary vertex of $G$. Then

$$\begin{aligned} D_{G,w}(\phi) &= \sum_{x \in V(G)} d_{G,w}(x, \phi(x)) \\ &\leq \sum_{x \in V(G)} (d_{G,w}(x, y) + d_{G,w}(\phi(x), y)) \\ &= \sum_{x \in V(G)} d_{G,w}(x, y) + \\ &\quad \sum_{x \in V(G)} d_{G,w}(\phi(x), y) \\ &= 2s_{G,w}(y). \end{aligned}$$

Since $\phi$ is an arbitrary permutation, and $y$ is an arbitrary vertex, we have

$$max\, D_{G,w}(\phi) \leq min\, 2s_{G,w}(y).$$

Thus $D(G, w) \leq 2s(G, w)$. □

Let $(G, w)$ be a weighted graph. A permutation $\pi$ of $V(G)$ is *optimal for the displacement* of $(G, w)$ if $D_{G,w}(\pi) = D(G, w)$. In Theorem 2.3 the min-max inequality about displacement and status will be shown to be an equality for some class of connected graphs, and the optimal permutations for the displacement are also characterized for these graphs. Let us begin with the following observation.

**Lemma 2.2** *Let $X_1, X_2, \cdots, X_m$ be disjoint nonempty subsets of a set $X$. If $|X_i| \leq \frac{1}{2}|X|$ for $i = 1, 2, \cdots, m$. Then there exists a permutation $\phi$ of $X$ such that $\phi(X_i) \cap X_i = \emptyset$ for $i = 1, 2, \cdots, m$.*

**Proof.** Let $X = \{x_1, x_2, \cdots, x_{|X|}\}$ such that

$$\begin{aligned} X_1 &= \{x_1, x_2, \cdots, x_{|X_1|}\}, \\ X_2 &= \{x_{|X_1|+1}, x_{|X_1|+2}, \cdots, x_{|X_1|+|X_2|}\}, \\ &\vdots \\ X_m &= \{x_{|X_1|+|X_2|+\cdots+|X_{m-1}|+1}, \\ &\quad x_{|X_1|+|X_2|+\cdots+|X_{m-1}|+2}, \cdots, \\ &\quad x_{|X_1|+|X_2|+\cdots+|X_m|}\}. \end{aligned}$$

Let $A$ be an integer such that $|X_i| \leq A \leq \frac{1}{2}|X|$ for $i = 1, 2, \cdots, m$.

Define $\phi : X \to X$ by $\pi(x_i) = x_{i+A}$ (the subscripts being taken modulo $|X|$) for $i = 1, 2, \cdots, |X|$. It is easy to see that $\phi$ satisfies the required property. □

**Theorem 2.3** *Suppose that $(G, w)$ is a connected weighted graph which has a cut vertex $v$ such that $|V(G')| \leq \frac{1}{2}|V(G)|$ for every component $G'$ of $G - v$. Then we have*
*(1) $D(G, w) = 2s(G, w)$,*
*(2) a permutation $\pi$ of $V(G)$ is optimal for the displacement of $(G, w)$ if and only if $d_{G,w}(x, \pi(x)) = d_{G,w}(x, v) + d_{G,w}(v, \pi(x))$ whenever the vertices $x$ and $\pi(x)$ are in the same component of $G - v$.*

**Proof.** We first show (1) and the sufficiency of (2). Since $|V(G')| \leq \frac{1}{2}|V(G)|$ for every component $G'$ of $G - v$, by Lemma 2.2 there exists a permutation $\phi$ of $V(G)$ such that $\phi(V(G')) \cap V(G') = \emptyset$ for every component $G'$ of $G - v$. Let $\pi$ be any permutation of $V(G)$ with this property $d_{G,w}(x, \pi(x)) = d_{G,w}(x, v) + d_{G,w}(v, \pi(x))$ whenever the vertices $x$ and $\pi(x)$ are in the same component of $G - v$. (The permutation $\phi$ satisfies this property trivially.) Then for every $x \in V(G)$, $d_{G,w}(x, \pi(x)) = d_{G,w}(x, v) + d_{G,w}(v, \pi(x))$. Hence

$$\begin{aligned} D_{G,w}(\pi) &= \sum_{x \in V(G)} d_{G,w}(x, \pi(x)) \\ &= \sum_{x \in V(G)} (d_{G,w}(x, v) + d_{G,w}(\pi(x), v)) \\ &= 2 \sum_{x \in V(G)} d_{G,w}(x, v) \\ &= 2s_{G,w}(v) \\ &\geq 2s(G, w). \end{aligned}$$

On the other hand, by Theorem 2.1,

$$2s(G, w) \geq D(G, w) \geq D_{G,w}(\pi).$$

Thus $D_{G,w}(\pi) = D(G, w) = 2s(G, w)$.

Hence we have proved (1) and the sufficiency of (2). Now we prove the necessity of (2). Suppose, on the contrary, there exists a vertex $z$ with $d_{G,w}(z, \pi(z)) < d_{G,w}(z, v) + d_{G,w}(v, \pi(z))$. Then

$$\begin{aligned} D_{G,w}(\pi) &= \sum_{x \in V(G)} d_{G,w}(x, \pi(x)) \\ &< \sum_{x \in V(G)} (d_{G,w}(x, v) + d_{G,w}(\pi(x), v)) \\ &= 2s_{G,w}(v) \\ &= 2s(G, w). \end{aligned}$$

The last equality follows from Theorem 1.2. By (1) of this theorem, we have $D_{G,w}(\pi) < D(G, w)$, a contradiction. This complete the proof. □

**Corollary 2.4** *If $(T, w)$ is a weighted tree, then*
*(1) $D(T, w) = 2s(T, w)$,*
*(2) a permutation $\pi$ of $V(T)$ is optimal for the displacement of $(T, w)$ if and only if $\pi(V(T')) \cap V(T') = \emptyset$ for every component $T'$ of $T - v$ where $v$ is in the median of $T$.*

**Proof.** This follows from Proposition 1.1, Lemma 2.2 and Theorem 2.3. □

An easy consequence follows from Corollary 2.4 (1).

**Corollary 2.5** *If $(T, w)$ is a tree with integral weights (i.e., $w(e)$ is an integer for each $e \in E(T)$), then $D(T, w)$ is an even integer.* □

Corollary 2.5 can also follow from the following result.

**Theorem 2.6** *Let $(T, w)$ be a weighted tree with integral weights. Suppose that $\pi$ is a permutation of $V(T)$. Then $D_{T,w}(\pi)$ is an even integer.*

**Proof.** For $y, z \in V(T)$, we use $[y, z]$ to denote the set of edges in the path which joins $y$ and $z$. We have

$$\begin{aligned} D_{T,w}(\pi) &= \sum_{x \in V(T)} d_{T,w}(x, \pi(x)) \\ &= \sum_{x \in V(T)} \sum_{e \in [x,\pi(x)]} w(e) \\ &= \sum_{e \in E(T)} \sum_{x : e \in [x,\pi(x)]} w(e) \\ &= \sum_{e \in E(T)} w(e) |\{x : e \in [x, \pi(x)]\}|. \end{aligned}$$

Let $e$ be an arbitrary edge in $T$, we show that $|\{x : e \in [x, \pi(x)]\}|$ is an even integer. Let $B_1$ and $B_2$ be the components of $T - e$. Then
$\{x : e \in [x, \pi(x)]\} =$
$\{x : x \in V(B_1), \pi(x) \in V(B_2)\} \bigcup$
$\{x : x \in V(B_2), \pi(x) \in V(B_1)\}$.
Since $|\{x : x \in V(B_1), \pi(x) \in V(B_2)\}| =$
$|\{x : x \in V(B_2), \pi(x) \in V(B_1)\}|$, we have that
$|\{x : e \in [x, \pi(x)]\}|$ is even. Thus $D_{T,w}(\pi)$ is even. □

## 3 Product Result for Displacement

Let $G$ and $H$ be graphs. The Cartesian product graph $G \times H$ of $G$ and $H$ is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{((x, y), (x', y')) : x = x' \in V(G), yy' = (y, y') \in E(H)$ *or* $xx' = (x, x') \in E(G), y = y' \in V(H)\}$.

Let $(G, w)$ and $(H, u)$ be weighted graphs. Then the Cartesian product graph of $(G, w)$ and $(H, u)$ is the weighted graph $(G \times H, w \times u)$ where the weight function $w \times u$ is defined as follows: if $((x, y), (x', y')) \in E(G \times H)$ then

$$w \times u((x, y), (x', y')) = \begin{cases} u(yy') & if \quad x = x', \\ w(xx') & if \quad y = y'. \end{cases}$$

It is easy to see that if $(x, y)$ and $(x', y')$ are vertices in $(G \times H, w \times u)$, then
$d_{G \times H, w \times u}((x, y), (x', y'))$
$= d_{G,w}(x, x') + d_{H,u}(y, y')$.

Now we will establish a product result about displacements of weighted graphs. We begin with a lemma.

**Lemma 3.1** *Let $X$ be the set $\{x_1, x_2, \cdots, x_m\}$, and $A$ be an $m$ by $n$ matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mi} & \cdots & a_{mn} \end{pmatrix}$$

*such that each element of $X$ appears $n$ times in $A$. Then for each $i$ $(i = 1, 2, \cdots, m)$, there exists a permutation $a_1^i a_2^i \cdots a_n^i$ of $a_{i1} a_{i2} \cdots a_{in}$ such that for each $j$ $(j = 1, 2, \cdots, n)$, $a_j^1, a_j^2, \cdots, a_j^m$ are distinct elements (i.e., $\{a_j^1, a_j^2, \cdots, a_j^m\} = \{x_1, x_2, \cdots, x_m\}$).*

**Proof.** First we choose $a_1^1, a_1^2, \cdots, a_1^m$ as follows: For $i = 1, 2, \cdots, m$, let $B_i = \{a_{ij} : 1 \leq j \leq n\}$. Let $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. Since each element in $X$ appears $n$ times in $A$, it appears at most $n$ times in the cells of $i_1$-th, $i_2$-th, $\cdots$, $i_k$-th rows of $A$. The $i_1$-th, $i_2$-th, $\cdots$, $i_k$-th rows has $kn$ cells. Thus $|B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_k}| \geq k$. By Hall's Theorem, there exist

$$a_{1j_1} \in B_1, a_{2j_2} \in B_2, \cdots, a_{mj_m} \in B_m$$

such that $a_{1j_1}, a_{2j_2}, \cdots, a_{mj_m}$ are distinct. Let

$$a_1^1 = a_{1j_1}, a_1^2 = a_{2j_2}, \cdots, a_1^m = a_{mj_m}.$$

Next we choose $a_2^1, a_2^2, \cdots, a_2^m$ as follows :
For $i = 1, 2, \cdots, m$, let $B_i{}' = \{a_{ij} : 1 \leq j \leq n, j \neq j_i\}$. For $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, by the same arguments as above, we have $|B'_{i_1} \cup B'_{i_2} \cup \cdots \cup B'_{i_k}| \geq k$. Again, by Hall's Theorem, there exist

$$a_{1j_1'} \in B_1', a_{2j_2'} \in B_2', \cdots, a_{mj_m'} \in B_m'$$

such that $a_{1j_1'}, a_{2j_2'}, \cdots, a_{mj_m'}$ are distinct. Let

$$a_2^1 = a_{1j_1'}, a_2^2 = a_{2j_2'}, \cdots, a_2^m = a_{mj_m'}.$$

Repeating the above procedure, we eventually have $a_j^i$ $1 \leq i \leq m, 1 \leq j \leq n$ such that $a_j^1, a_j^2, \cdots, a_j^m$ are distinct for every $j$ $(j = 1, 2, \cdots, n)$. We can also see that $a_1^i a_2^i \cdots a_n^i$ is a permutation of $a_{i1} a_{i2} \cdots a_{in}$ for each $i (i = 1, 2, \cdots, m)$. □

Now we prove the product result for displacements.

**Theorem 3.2** *Suppose that $(G, w), (H, u)$ are connected weighted graphs, and the orders of $G$ and $H$ are $m$ and $n$ respectively. Then*

$$D(G \times H, w \times u) = nD(G, w) + mD(H, u).$$

**Proof.** First we prove that $D(G \times H, w \times u) \geq nD(G,w) + mD(H,u)$. Let $\pi_1$ be a permutation of $V(G)$ such that $D_{G,w}(\pi_1) = D(G,w)$. Let $\pi_2$ be a permutation of $V(H)$ such that $D_{H,u}(\pi_2) = D(H,u)$. Define $\pi : V(G \times H) \to V(G \times H)$ by $\pi(x,y) = (\pi_1(x), \pi_2(y))$. Then

$$\begin{aligned}
& D_{G\times H, w\times u}(\pi) \\
= & \sum_{(x,y)\in V(G\times H)} d_{G\times H,w\times u}((x,y),\pi(x,y)) \\
= & \sum_{(x,y)\in V(G\times H)} d_{G\times H,w\times u}((x,y),(\pi_1(x),\pi_2(y))) \\
= & \sum_{(x,y)\in V(G\times H)} (d_{G,w}(x,\pi_1(x)) + d_{H,u}(y,\pi_2(y))) \\
= & \sum_{(x,y)\in V(G\times H)} d_{G,w}(x,\pi_1(x)) \\
& + \sum_{(x,y)\in V(G\times H)} d_{H,u}(y,\pi_2(y)) \\
= & nD_{G,w}(\pi_1) + mD_{H,u}(\pi_2) \\
= & nD(G,w) + mD(H,u).
\end{aligned}$$

Since $D(G \times H, w \times u) \geq D_{G\times H,w\times u}(\pi)$, we have $D(G \times H, w \times u) \geq n\, D(G,w) + m\, D(H,u)$.

Next we prove the reverse inequality $D(G \times H, w \times u) \leq n\, D(G,w) + m\, D(H,u)$. Let $\pi$ be an arbitrary permutation of $V(G\times H)$. We will show that $D_{G\times H,w\times u}(\pi) \leq n\, D(G,w) + m\, D(H,u)$. Let $p_1$ be the function from $V(G\times H)$ to $V(G)$ such that $p_1(x,y) = x$ for $(x,y) \in V(G \times H)$, and $p_2$ be the function from $V(G\times H)$ to $V(H)$ such that $p_2(x,y) = y$ for $(x,y) \in V(G\times H)$. Note

$$\begin{aligned}
& d_{G\times H,w\times u}((x,y),\pi(x,y)) \\
= & d_{G\times H,w\times u}((x,y),(p_1(\pi(x,y)),p_2(\pi(x,y)))) \\
= & d_{G,w}(x,p_1(\pi(x,y))) + d_{H,u}(y,p_2(\pi(x,y))).
\end{aligned}$$

Thus

$$\begin{aligned}
& D_{G\times H,w\times u}(\pi) \\
= & \sum_{(x,y)\in V(G\times H)} d_{G\times H,w\times u}((x,y),\pi(x,y)) \\
= & \sum_{(x,y)\in V(G\times H)} d_{G,w}(x,p_1(\pi(x,y))) \\
& + \sum_{(x,y)\in V(G\times H)} d_{H,u}(y,p_2(\pi(x,y))).
\end{aligned}$$

Let

$$D_1 = \sum_{(x,y)\in V(G\times H)} d_{G,w}(x,p_1(\pi(x,y))),$$

$$D_2 = \sum_{(x,y)\in V(G\times H)} d_{H,u}(y,p_2(\pi(x,y))).$$

Here we show that $D_1 \leq n\, D(G,w)$. Let $V(G) = \{x_1, x_2, \cdots, x_m\}$, $V(H) = \{y_1, y_2, \cdots, y_n\}$. Since $\pi$ is a permutation of $V(G\times H)$, we have $\{\pi(x_i,y_j) : 1 \leq i \leq m, 1 \leq j \leq n\} = \{(x_i,y_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. Thus each $x_i (1 \leq i \leq m)$ appears $n$ times in the array

$$(p_1(\pi(x_i,y_j)))_{1\leq i\leq m, 1\leq j\leq n}.$$

Let $a_{ij} = p_1(\pi(x_i,y_j)), 1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 3.1, for each $i$ $(i = 1,2,\cdots,m)$, there exists a permutation $a_1^i a_2^i \cdots a_n^i$ of $a_{i1}a_{i2}\cdots a_{in}$ such that for each $j$ $(j = 1,2,\cdots,n)$, $\{a_j^1, a_j^2, \cdots a_j^m\} = \{x_1, x_2, \cdots x_m\}$. Thus

$$\begin{aligned}
D_1 &= \sum_{1\leq i\leq m}\sum_{1\leq j\leq n} d_{G,w}(x_i,p_1(\pi(x_i,y_j))) \\
&= \sum_{1\leq i\leq m}\sum_{1\leq j\leq n} d_{G,w}(x_i,a_{ij}) \\
&= \sum_{1\leq i\leq m}\sum_{1\leq j\leq n} d_{G,w}(x_i,a_j^i) \\
&= \sum_{1\leq j\leq n}\sum_{1\leq i\leq m} d_{G,w}(x_i,a_j^i) \\
&\leq \sum_{1\leq j\leq n} D(G,w) \\
&= n\, D(G,w).
\end{aligned}$$

Similarly, we can show that $D_2 \leq m\, D(H,u)$. Thus

$$\begin{aligned}
D_{G\times H,w\times u}(\pi) &= D_1 + D_2 \\
&\leq n\, D(G,w) + m\, D(H,u).
\end{aligned}$$

Since $\pi$ is an arbitrary permutation of $V(G\times H)$, we obtain

$$D(G\times H, w\times u) \leq n\, D(G,w) + m\, D(H,u).$$

This completes the proof. □

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