Adjacent vertex fault tolerance for two spanning disjoint paths of hypercube

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Abstract

Let F_a be the set of $|F_a|$ pairs of adjacently faulty vertices and F_e be the set of faulty edges of Q_n. Let s_1, s_2, t_1, t_2 be four distinct fault-free vertices of $Q_n = (V_b \cup V_w, E)$ for $s_1, t_1 \in V_b$ and $s_2, t_2 \in V_w$. There exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n - F_a - F_e$ with $|F_a| \leq n-4$ and $|F_a| + |F_e| \leq n-3$. Let $F_v \subset V_b$ be a set of two faulty vertices of Q_n. Let $s_1, s_2, t_1, t_2 \in V_w$ be four distinct fault-free vertices of Q_n. There exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n - F_a - F_v$ with $|F_a| \leq n-4$.

1 Introduction

The hypercube network is the most popular and efficient topological structures of interconnection networks. This is partially due to its many excellent features, such as regularity, symmetry, low diameter and degree, high connectivity, simple routing algorithms, and so on. Linear array and rings are two of the most fundamental interconnection networks. They are suitable for designing simply algorithms with low costs. Component failures are unavoidable in a large parallel systems. Thus, fault tolerance is an important research issue for interconnection networks.

An interconnection network is usually represented by a graph where vertices represent processors and edges represent links between processors. Let $G = (V_b \cup V_w, E)$ be a *bipartite graph* where V_b and V_w are two disjoint vertex sets such that each edge of E consists of one vertex from each set. A bipartite graph $G = (V_b \cup V_w, E)$ is *Hamiltonian laceable* if there exists a *Hamiltonian path* between x, y for any $x \in V_b, y \in V_w$. The graph $G = (V_b \cup V_w, E)$ is *hyper-Hamiltonian laceable* if $\forall v \in V_b(resp. V_w)$, there exists a Hamiltonian path of $G - \{v\}$ between each pair of vertices of $V_w(resp. V_b)$. In [3], Tsai et al. proved that $Q_n - F_e$ is Hamiltonian laceable for $F_e \subset E$ and $|F_e| \leq n-2$ and $Q_n - F_e$ is hyper-Hamiltonian laceable for $F_e \subset E$ and $|F_e| \leq n-3$.

In [1], Hung et al. studied the adjacent vertices fault tolerance Hamiltonian laceability for hypercube. Let F_a be the set of adjacently faulty vertices of Q_n and $|F_a|$ be the number of pairs of F_a. That is, F_a can be denoted as $\{a_i, b_i | a_i \in V_b, b_i \in V_w \text{ and } (a_i, b_i) \in E \text{ for } 1 \leq i \leq |F_a|\}$. A bipartite graph $G = (V_b \cup V_w, E)$ is *f-adjacency k edges Hamiltonian* if $G - F_a$ is k edges Hamiltonian for $|F_a| = f$. A bipartite graph $G = (V_b \cup V_w, E)$ is *f-adjacency k edges Hamiltonian laceable* if $G - F_a$ is k edges Hamiltonian laceable for $|F_a| = f$. A bipartite graph $G = (V_b \cup V_w, E)$ is *f-adjacency k edges hyper-Hamiltonian laceable* if $G - F_v$ is k edges hyper-Hamiltonian laceable for $|F_a| = f$. In [1], the authors proved that Q_n is f-adjacency $(n-2-f)$ edges Hamiltonian for $f \leq n-2$, f-adjacency $(n-2-f)$ edges Hamiltonian laceable for $f \leq n-3$, and f-adjacency $(n-3-f)$ edges hyper-Hamiltonian laceable for $f \leq n-3$.

Let P_1 and P_2 be two paths of a graph G. The two paths P_1 and P_2 are *two spanning disjoint paths* if every vertex of G is either on P_1 or on P_2. A bipartite graph $G = (V_b \cup V_w, E)$ has *property 2H with different color* if for any $s_1, s_2 \in V_b, t_1, t_2 \in V_w$ there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of G. A bipartite graph G is k edges property 2H with different color if $G - F_e$ has property 2H with different color where F_e is the set of edges of G. In [1], the authors proved that Q_n is $(n-3)$ edges property 2H with different color for $n \geq 3$. A bipartite graph G is f-adjacency k edges property 2H with different color if $G - F_a$ is k edges property 2H

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with different color for $|F_a| = f$. Su et al. showed that Q_n is f-adjacency $(n-3-f)$ edges property 2H with different color for $f \leq n-3$ in [2].

In this paper, we will investigate adjacency fault tolerance for some other property 2H. A bipartite graph $G = (V_b \cup V_w, E)$ has *property 2H with same color* if for any $s_1, t_1 \in V_b, s_2, t_2 \in V_w$ there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of G. A bipartite graph G is f-adjacency k edges property 2H with same color if $G - F_a - F_e$ has property 2H with same color for $|F_a| = f$ and $|F_e| = k$. A bipartite graph $G = (V_b \cup V_w, E)$ has *property 2H adding two black faulty nodes* if for any $f_1, f_2 \in V_b$ and $s_1, t_1, s_2, t_2 \in V_w$ there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $G - \{f_1, f_2\}$. A bipartite graph G is f-adjacency property 2H adding two black faulty nodes if $G - F_a$ has property 2H adding two black faulty nodes for $|F_a| = f$. In this paper, we show that Q_n is f-adjacency $(n-3-f)$ edges property 2H with same color for $f \leq n-4$ and Q_n is f-adjacency property 2H adding two black faulty nodes for $f \leq n-4$. These results are optimal.

2 Definitions and Notations

An n-dimensional hypercube $Q_n = (V_b \cup V_w, E)$ is a bipartite graph whose vertices are labeled by distinct n-bit binary strings. Two vertices are linked by an edge if and only if their labels differ exactly in one bit. The hypercube Q_n can be constructed recursively as $Q_n = Q_n \times K_2$. We can partition Q_n as two subgraphs Q_n^0 and Q_n^1 by choosing any one bit of binary string.

Let F_a be the set of adjacently faulty vertices of Q_n. We use F_a^j to denote the adjacently faulty vertex set of Q_n^j, respectively, for $j = 0, 1$. Thus, $F_a = F_a^0 \cup F_a^1$. We also use $|F_a^j|$ to denote the number of adjacently pairs of F_a^j, for $j = 0, 1$. Let F_e be the set of faulty edges of Q_n.

We use F_e^j to denote the set of faulty edges of Q_n^j, respectively, for $j = 0, 1$. Thus, $F_e = F_e^0 \cup F_e^1$. Let $F_v^j \subset V_b$ to denote the set of two faulty vertices of Q_n^j, respectively, for $j = 0, 1$. Thus, $F_v = F_v^0 \cup F_v^1$. Let $U = \{s_1, s_2, t_1, t_2\}$ and $U^i = U \cap V(Q_n^i)$ for $i = 0, 1$. Let x be a vertex of Q_n^i, x' be the vertex of Q_n^j such that $(x, x') \in E$ for $\{i, j\} = \{0, 1\}$.

The following lemma is proposed in [1].

Lemma 1 *The graph Q_n is f-adjacency $(n-2-f)$ edges Hamiltonian for $0 \leq f \leq (n-2)$, f-adjacency $(n-2-f)$ edges Hamiltonian laceable for $0 \leq f \leq (n-3)$, and f-adjacency $(n-3-f)$ edges hyper-Hamiltonian laceable for $0 \leq f \leq (n-3)$.*

The following lemmas are proved in [2].

Lemma 2 *The graph Q_n is f-adjacency $(n-3-f)$ edges property 2H, for $0 \leq f \leq n-3, n \geq 3$.*

Lemma 3 *Let $Q_n = (V_b \cup V_w, E)$. For $a \in V_b, b \in V_w$, the graph $Q_n - \{a, b\}$ is f-adjacency $(n-f-3)$ edges Hamiltonain laceable for $0 \leq f \leq n-4$, $n \geq 4$.*

3 Two spanning disjoint paths of hypercube

Since Q_n^0 is Hamiltonian laceable, there exists a Hamiltonian path $P(x_0, y_0)$ of Q_n^0. We can denote $P(x_0, y_0)$ as $\langle x_0 \overset{P(x_0,w_0)}{\longrightarrow} w_0, w_1 \overset{P(w_1,y_0)}{\longrightarrow} y_0 \rangle$, $w_0 \in V_w$, $w_1 \in V_b$ and $\phi(w_0) \neq y_1$, $\phi(w_1) \neq x_1$.

Theorem 1 *The graph Q_n is f-adjacency $(n-3-f)$ edges property 2H with same color for $f \leq n-4$.*

Proof. We will prove this theorem by induction on n. The graph Q_4 can be verified to be true by brute force. Let F_a be the set of $|F_a|$ pairs of adjacently faulty vertices and F_e be the set of faulty edges of Q_n. With the symmetry of hypercube, we can assume that every faulty pair of F_a and every faulty edge is in the some subcube. Let s_1, s_2, t_1, t_2 be four distinct fault-free vertices of $Q_n = (V_b \cup V_w, E)$ for $s_1, t_1 \in V_b$ and $s_2, t_2 \in V_w$. We will construct two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n - F_a - F_e$ with $|F_a| \leq n-4$ and $|F_a| + |F_e| \leq n-3$.

In the follows, we will prove this theorem holds for $n \geq 5$.

Case 1 $F_a^i \cup F_e^i = \emptyset$ for some $i = 0, 1$. Without loss of generality, we can assume that $F_a^1 \cup F_e^1 = \emptyset$.

Case 1.1 $|U^0| = 4$.
Applying Lemma 1, we can construct a Hamiltonian path $P(s_2, t_1)$ of $Q_n^0 - F_a - F_e$. Without loss of generality, we can assume that $P(s_2, t_1) = \langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, s_1 \overset{P(s_1,b_1)}{\longrightarrow} b_1, t_2, b_2 \overset{P(b_2,t_1)}{\longrightarrow} t_1 \rangle$. With the induction hypothesis, there exist two spanning disjoint paths $P(w_1', t_2')$ and $P(b_1', b_2')$ of Q_n^1. Thus, $\langle s_1 \overset{P(s_1,b_1)}{\longrightarrow}$

$b_1, b'_1 \stackrel{P(b'_1, b'_2)}{\longrightarrow} b'_2, b_2 \stackrel{P(b_2,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \stackrel{P(s_2,w_1)}{\longrightarrow} w_1, w'_1 \stackrel{P(w'_1,t'_2)}{\longrightarrow} t'_2, t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_e$, as illustrated in Figure 1.

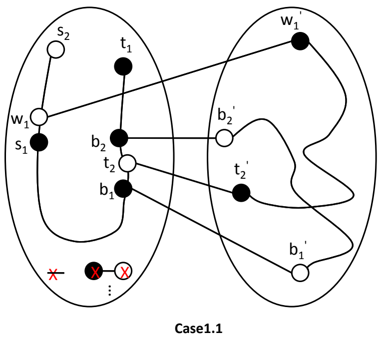

Figure 1: Illustration of case 1.1 of Theorem 1.

Case 1.2 $|U^0| = 3$.
Without loss of generality, we can assume that $t_1 \in Q_n^1$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \stackrel{P(s_1,b_1)}{\longrightarrow} b_1, s_2 \stackrel{P(s_2,t_2)}{\longrightarrow} t_2\rangle$ of $Q_n^0 - F_a - F_e$. We can also construct a Hamiltonian path $P(b'_1, t_1)$ of Q_n^1. Thus, $\langle s_1 \stackrel{P(s_1,b_1)}{\longrightarrow} b_1, b'_1 \stackrel{P(b'_1,t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 1.3 $|U^0| = 2$ and the vertices in U^0 with different color.
Without loss of generality, we can assume that $t_1, t_2 \in Q_n^1$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \stackrel{P(s_1,b_1)}{\longrightarrow} b_1, w_1 \stackrel{P(w_1,s_2)}{\longrightarrow} s_2\rangle$ of $Q_n^0 - F_a - F_e$. Applying Lemma 2, we can construct two spanning disjoint paths $P(b'_1, t_1)$ and $P(w'_1, t_2)$ of Q_n^1. Thus, $\langle s_1 \stackrel{P(s_1,b_1)}{\longrightarrow} b_1, b'_1 \stackrel{P(b'_1,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \stackrel{P(s_2,w_1)}{\longrightarrow} w_1, w'_1 \stackrel{P(w'_1,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 1.4 $|U^0| = 2$ and the vertices in U^0 with the same color.
Without loss of generality, we can assume that $s_2, t_2 \in Q_n^1$. Let w_1 be a fault-free vertex in Q_n^0 with $d(w_1, t_1) \geq 3$ and$w_1 \in V_w$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \stackrel{P(s_1,t_1)}{\longrightarrow} t_1, w_2 \stackrel{P(w_2,w_1)}{\longrightarrow} w_1\rangle$ of $Q_n^0 - F_a - F_e$. Applying Lemma 2, we can construct two spanning disjoint paths $P(w'_1, s_2)$ and $P(w'_2, t_2)$ of Q_n^1. Thus, $P(s_1, t_1)$ and $\langle s_2 \stackrel{P(s_2,w'_1)}{\longrightarrow} w'_1, w_1 \stackrel{P(w_1,w_2)}{\longrightarrow} w_2, w'_2 \stackrel{P(w'_2,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 1.5 $|U^0| = 1$.
Without loss of generality, we can assume that $t_1, s_2, t_2 \in Q_n^1$. Let w_1 be a fault-free vertex in Q_n^0 with $w'_1 \neq t_1$ and $w_1 \in V_w$. With the induction hypothesis, there exist two spanning disjoint paths $P(t_1, w_1)$ and $P(s_2, t_2)$ of Q_n^1. Thus, $\langle s_1 \stackrel{P(s_1,w_1)}{\longrightarrow} w_1, w'_1 \stackrel{P(w'_1,t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 1.6 $|U^0| = 0$.
With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of Q_n^1. Since $2^n - 2 > 2 \cdot 2 \cdot |F_a|$, there exists an edge (w_1, b_1) of $P(s_1, t_1)$ or $P(s_2, t_2)$ with $w'_1, b'_1 \notin F_a$ and $w_1 \in V_w$, $b_1 \in V_b$. Without loss of generality, we can assume that $P(s_1, t_1) = \langle s_1 \stackrel{P(s_1,w_1)}{\longrightarrow} b_1 \stackrel{P(b_1,t_1)}{\longrightarrow} t_1\rangle$. Applying Lemma 1, we can construct a Hamiltonian path $P(w'_1, b'_1)$ of $Q_n^0 - F_a - F_e$. Thus, $\langle s_1 \stackrel{P(s_1,w_1)}{\longrightarrow} w_1, w'_1 \stackrel{P(w'_1,b'_1)}{\longrightarrow} b'_1, b_1 \stackrel{P(b_1,t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 2 $F_a^i \cup F_e^i \neq \emptyset$ for $i = 0, 1$. Without loss of generality, we can assume that $|U^0| \geq |U^1|$.

Case 2.1 $|U^0| = 4$.
With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n^0 - F_a - F_e$. Since $2^n - 2 > 2 \cdot 2 \cdot |F_a|$, there exists an edge (b_1, w_1) of $P(s_1, t_1)$ or $P(s_2, t_2)$ with $b'_1, w'_1 \notin F_a$ and $b_1 \in V_b$, $w_1 \in V_w$. Without loss of generality, we can assume that $P(s_2, t_2) = \langle s_2 \stackrel{P(s_2,b_1)}{\longrightarrow} b_1, w_1 \stackrel{P(w_1,t_2)}{\longrightarrow} t_2\rangle$. Applying Lemma 1, we can construct a Hamiltonian path $P(w'_1, b'_1)$ of $Q_n^1 - F_a - F_e$. Thus, $P(s_1, t_1)$ and $\langle s_2 \stackrel{P(s_2,b_1)}{\longrightarrow} b_1, b'_1 \stackrel{P(b'_1,w'_1)}{\longrightarrow} w'_1, w_1 \stackrel{P(w_1,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 2.2 $|U^0| = 3$.

Without loss of generality, we can assume that $t_1 \in Q_n^1$. Let b_1 be a fault-free vertex in Q_n^0 with $b_1' \notin F_a$ and $b_1 \in V_b$. With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, b_1)$ and $P(s_2, t_2)$ of $Q_n^0 - F_a - F_e$. Applying Lemma 1, we can construct a Hamiltonian path $P(t_1, b_1')$ of $Q_n^1 - F_a - F_e$. Thus, $\langle s_1 \overset{P(s_1,b_1)}{\longrightarrow} b_1, b_1' \overset{P(b_1',t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 2.3 $|U^0| = 2$ and the vertices in U^0 with different color.

Without loss of generality, we can assume that $t_1, t_2 \in Q_n^1$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, t_1)$ of $Q_n^0 - F_a - F_e$. There exists an edge (b_1, w_1) of $P(s_1, t_1)$ with $b_1', w_1' \notin F_a$ and $b_1 \in V_b$, $w_1 \in V_w$. Applying Lemma 2, we can construct two spanning disjoint paths $P(b_1', t_1)$ and $P(w_1', t_2)$ of $Q_n^1 - F_a - F_e$. Thus, $\langle s_1 \overset{P(s_1,b_1)}{\longrightarrow} b_1, b_1' \overset{P(b_1',t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

Case 2.4 $|U^0| = 2$ and the vertices in U^0 with the same color.

Without loss of generality, we can assume that $s_2, t_2 \in Q_n^1$. Let w_1, w_2 be a fault-free vertex in Q_n^0 with $w_1', w_2' \notin F_a$ and $w_1, w_2 \in V_w$. With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(w_1, w_2)$ of $Q_n^0 - F_a - F_e$. Thus, $P(s_1, t_1)$ and $\langle s_2 \overset{P(s_2,w_1')}{\longrightarrow} w_1', w_1 \overset{P(w_1,w_2)}{\longrightarrow} w_2, w'2 \overset{P(w_2',t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_e$.

□

We will claim that the result of Theorem 1 is optimal. Let (x, y) be an edge of Q_n and $(x, x_i), (y, y_i)$ be edges of Q_n for $1 \le i \le n-1$. There are not two spanning disjoint paths $P(x_1, x_2)$ and $P(y_1, y_2)$ in $Q_n - \{x_j, y_j | \text{ for } 3 \le j \le n-1\}$. Therefore, Q_n is not $(n-3)$-adjacency property 2H with same color.

Theorem 2 *The graph Q_n is $(n-4)$-adjacency propery 2H adding two black faulty nodes for $n \ge 4$.*

Proof. We will prove this theorem by induction on n. The graph Q_4 can be verified to be true by brute force. Let F_a be the set of $|F_a|$ pairs of adjacently faulty vertices. Let f_x, f_y be an element of F_a with $f_x \in V_b, f_y \in V_w$ and $F_{aa} = F_a - \{f_x, f_y\}$. Let $F_v = \{f_1, f_2\}$ and $F_v^i = F_v \cap Q_n^i$ for $i = 0, 1$. Let $s_1, s_2, t_1, t_2 \in V_w$ be four distinct fault-free vertices of Q_n. With the symmetry of hypercube, we can assume that every pair of adjacently faulty vertices of F_a is in the some sub-cube. We will construct two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n - F_a - F_v$ with $|F_a| \le n-4$ for $n \ge 5$ with the following cases.

Case 1 $F_a^i \cup F_v^i = \emptyset$ for some $i = 0, 1$.

Without loss of generality, we can assume that $F_a^1 \cup F_v^1 = \emptyset$.

Case 1.1 $|U^0| = 4$.

With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n^0 - F_{aa} - F_v$. Suppose that f_x and f_y are in the same path $P(s_i, t_i)$ for some $i = 0, 1$. Without loss of generality, we can assume that $P(s_2, t_2) = \langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, f_x, w_2 \overset{P(w_2,b_1)}{\longrightarrow} b_1, f_y, b_2 \overset{P(b_2,t_2)}{\longrightarrow} t_2\rangle$. Applying Theorem 1, we can construct two spanning disjoint paths $P(w_1', w_2')$ and $P(b_1', b_2')$ of Q_n^1. Thus, $P(s_1, t_1)$ and $\langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,b_1)}{\longrightarrow} b_1, b'1 \overset{P(b_1',b_2')}{\longrightarrow} b_2', b_2 \overset{P(b_2,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$, as illustrated in Figure 2.

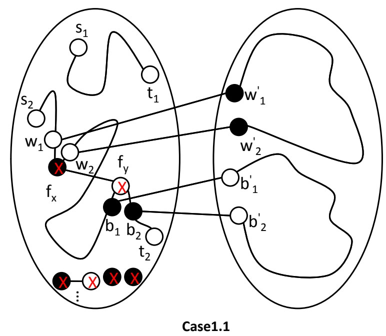

Figure 2: Illustration of case 1.1 of Theorem 2.

Suppose that f_x and f_y are in different path $P(s_i, t_i)$ for some $i = 0, 1$.

Without loss of generality, we can assume that $P(s_1,t_1) = \langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, f_x, w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2,t_2) = \langle s_2 \overset{P(s_2,b_1)}{\longrightarrow} b_1, f_y, b_2 \overset{P(b_2,t_2)}{\longrightarrow} t_2\rangle$. And there exist two spanning disjoint paths $P(w_1',w_2')$ and $P(b_1',b_2')$ of Q_n^1. Thus, $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,b_1)}{\longrightarrow} b_1, b'1 \overset{P(b_1',b_2')}{\longrightarrow} b_2', b_2 \overset{P(b_2,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 1.2 $|U^0| = 3$.

Without loss of generality, we can assume that $t_2 \in Q_n^1$. With the induction hypothesis, there exist two spanning disjoint paths $P(s_1,t_1)$ and $P(s_2,f_y)$ of $Q_n^0 - F_{aa} - F_v$. Suppose that f_x in $P(s_2,f_y)$. Without loss of generality, we can assume that $P(s_2,f_y) = \langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, f_x, w_2 \overset{P(w_2,b_1)}{\longrightarrow} b_1, f_y\rangle$. Suppose that $b_1' \neq t_2$. Applying Theorem 1, we can construct two spanning disjoint paths $P(w_1',w_2')$ and $P(b_1',t_2)$ of Q_n^1. Thus, $P(s_1,t_1)$ and $\langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,b_1)}{\longrightarrow} b_1, b_1' \overset{P(b_1',t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$. When $b_1' = t_2$, there exists a Hamiltonian path $P(w_1',w_2')$ of $Q_n^1 - t_2$. Thus, $P(s_1,t_1)$ and $\langle s_2 \overset{P(s_2,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,b_1)}{\longrightarrow} b_1, t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Suppose that f_x in $P(s_1,t_1)$. We can denote $P(s_1,t_1) = \langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, f_x, w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2,f_y) = \langle s_2 \overset{P(s_2,b_1)}{\longrightarrow} b_1, f_y\rangle$. Suppose that $b_1' \neq t_2$. Applying Theorem 1, we can construct two spanning disjoint paths $P(w_1',w_2')$ and $P(b_1',t_2)$ of Q_n^1. Thus, $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,b_1)}{\longrightarrow} b_1, b_1' \overset{P(b_1',t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$. When $b_1' = t_2$, there exists a Hamiltonian path $P(w_1',w_2')$ of $Q_n^1 - t_2$. Thus, $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,b_1)}{\longrightarrow} b_1, t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 1.3 $|U^0| = 2$ and the two vertices of U^0 are not the same pair.

Without loss of generality, we can assume that $t_1, t_2 \in Q_n^1$. Applying Lemma 2, we can construct two spanning disjoint paths $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, f_1\rangle$ and $P\langle s_2 \overset{P(s_2,w_2)}{\longrightarrow} w_2, f_2\rangle$ of $Q_n^0 - F_a$. We can also construct two spanning disjoint paths $P(w_1',t_1)$ and $P(w_2',t_2)$ of Q_n^1. Thus, $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \overset{P(s_2,w_2)}{\longrightarrow} w_2, w_2' \overset{P(w_2',t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 1.4 $|U^0| = 2$ and the two vertices of U^0 are the same pair.

Without loss of generality, we can assume that $s_2, t_2 \in Q_n^1$. Applying Lemma 2, we can construct two spanning disjoint paths $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, f_1\rangle$ and $P\langle f_2, w_2, t_1 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ of $Q_n^0 - F_a$. We can also construct two spanning disjoint paths $P(w_1',w_2')$ and $P(s_2,t_2)$ of Q_n^1. Thus, $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',w_2')}{\longrightarrow} w_2', w_2 \overset{P(w_2,t_1)}{\longrightarrow} t_1\rangle$ and $P(s_2,t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 1.5 $|U^0| = 1$.

Without loss of generality, we can assume that $t_1, s_2, t_2 \in Q_n^1$. Let (s_2, b_1) be a fault-free edge of Q_n^1 with $b_1' \notin F_a \cup \{s_1\}$. Applying Lemma 2, we can construct two spanning disjoint paths $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, f_1\rangle$ and $\langle b_1' \overset{P(b_1',w_2)}{\longrightarrow} w_2, f_2\rangle$ of $Q_n^0 - F_a$. We can also construct two spanning disjoint paths $P(w_1',t_1)$ and $P(w_2',t_2)$ of $Q_n^1 - \{s_2, b_1\}$. Thus, $\langle s_1 \overset{P(s_1,w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2, b_1, b_1' \overset{P(b_1',w_2)}{\longrightarrow} w_2, w_2' \overset{P(w_2',t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 1.6 $|U^0| = 0$.

When $n \leq 5$, we can redivide the Q_n with $|U^0|, |U^1| \geq 1$. We will prove this case with $n \geq 6$. Let (s_1, b_1) and (s_2, b_2) be two fault-free edges in Q_n^1 with $b_1', b_2' \notin F_a$. Applying Lemma 2, we can construct two spanning disjoint paths $\langle b_1' \overset{P(b_1',w_1)}{\longrightarrow} w_1, f_1\rangle$ and $\langle b_2' \overset{P(b_2',w_2)}{\longrightarrow}$

$w_2, f_2\rangle$ of $Q_n^0 - F_a$. We can also construct two spanning disjoint paths $P(w_1', t_1)$ and $P(w_2', t_2)$ of $Q_n^1 - \{s_1, b_1, s_2, b_2\}$ for $n \geq 6$. Thus, $\langle s_1, b_1, b_1' \xrightarrow{P(b_1', w_1)} w_1, w_1' \xrightarrow{P(w_1', t_1)} t_1\rangle$ and $\langle s_2, b_2, b_2' \xrightarrow{P(b_2', w_2)} w_2, w_2' \xrightarrow{P(w_2', t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$, as illustrated in Figure 3.

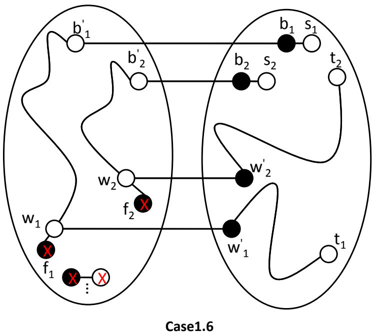

Figure 3: Illustration of case 1.6 of Theorem 2.

Case 2 $F_a^i = \emptyset$ and $|F_v^i| = 1$ for some $i = 0, 1$.
Without loss of generality, we can assume that $i = 0$.

Case 2.1 $|U^0| = 4$.
Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, t_1)$. Without loss of generality, we can assume that $P(s_1, t_1) = \langle s_1 \xrightarrow{P(s_1, b_1)} b_1, s_2 \xrightarrow{P(s_2, t_2)} t_2, b_2 \xrightarrow{P(b_2, t_1)} t_1\rangle$ of $Q_n^0 - F_a - F_v^0$. We can also construct a Hamiltonian path $P(b_1', b_2')$ of $Q_n^1 - F_v^1$. Thus, $\langle s_1 \xrightarrow{P(s_1, b_1)} b_1, b_1' \xrightarrow{P(b_1', b_2')} b_2', b_2 \xrightarrow{P(b_2, t_1)} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 2.2 $|U^0| = 3$.
Without loss of generality, we can assume that $t_2 \in Q_n^1$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, s_2) = P\langle s_1 \xrightarrow{P(s_1, t_1)} t_1, b_1 \xrightarrow{P(b_1, s_2)} s_2\rangle$ of $Q_n^0 - F_a - F_v^0$.
Suppose that $b_1' \neq t_2$. We can also construct a Hamiltonian path $P(b_1', t_2)$ of $Q_n^1 - F_v^1$. Thus, $P(s_1, t_1)$ and $P\langle s_2 \xrightarrow{P(s_2, b_1)} b_1, b_1' \xrightarrow{P(b_1', t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Suppose that $b_1' = t_2$. Since $2^n - 2 > 2 \cdot 2 \cdot |F_a|$, there exists an edge (w_1, b_1) of $P(s_1, s_2)$ with $w_1', b_1' \notin F_v$ and $w_1 \in V_w$, $b_1 \in V_b$. Without loss of generality, we can assume that $P(s_1, s_2) = \langle s_1 \xrightarrow{P(s_1, w_1)} w_1, b_2 \xrightarrow{P(b_2, t_1)} t_1, b_1 \xrightarrow{P(b_1, s_2)} s_2\rangle$. Applying Lemma 3, we can construct a Hamiltonian path $P(w_1', b_2')$ of $Q_n^1 - t_2 - F_v^1$. Thus, $\langle s_1 \xrightarrow{P(s_1, w_1)} w_1, w_1' \xrightarrow{P(w_1', b_2')} b_2', b_2 \xrightarrow{P(b_2, t_1)} t_1\rangle$ and $\langle s_2 \xrightarrow{P(s_2, b_1)} b_1, t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$, as illustrated in Figure 4.

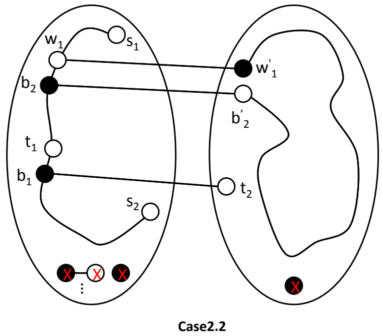

Figure 4: Illustration of case 2.2 of Theorem 2.

Case 2.3 $|U^0| = 2$ and the two vertices of U^0 are not the same pair.
Without loss of generality, we can assume that $t_1, t_2 \in Q_n^1$. Applying Lemma 1, we can construct a Hamiltonian path $\langle s_1 \xrightarrow{P(s_1, w_1)} w_1, b_1 \xrightarrow{P(b_1, s_2)} s_2\rangle$ of $Q_n^0 - F_a - F_v^0$ with $w_1', b_1' \notin (F_v^1 \cup U^1)$. Let (w_1', x) be a fault-free edge of Q_n^1. With the induction hypothesis, there exist two spanning disjoint paths $P(x, t_1)$ and $P(b_1', t_2)$ of $Q_n^1 - F_v - \{w_1'\}$. Thus, $\langle s_1 \xrightarrow{P(s_1, w_1)} w_1, w_1', x \xrightarrow{P(x, t_1)} t_1\rangle$ and $\langle s_2 \xrightarrow{P(s_2, b_1)} b_1, b_1' \xrightarrow{P(b_1', t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 2.4 $|U^0| = 2$ and the two vertices of U^0 are the same pair.
Without loss of generality, we can assume that $s_2, t_2 \in Q_n^1$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_i, t_i)$ of $Q_n^i - F_a^i - F_v^i$ for $i = 0, 1$. Thus, $P(s_1, t_1)$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 2.5 $|U^0| = 1$.
Without loss of generality, we can as-

sume that $t_1, s_2, t_2 \in Q_n^1$. Let w_1 be a fault-free vertex of in Q_n^0 with $w_1' \neq F_a$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, w_1)$ of $Q_n^0 - F_a - F_v^0$. With the induction hypothesis, there exist two spanning disjoint paths $P(t_1, w_1')$ and $P(s_2, t_2)$ of $Q_n^1 - F_v^1$. Thus, $\langle s_1 \xrightarrow{P(s_1,w_1)} w_1, w_1' \xrightarrow{P(w_1',t_1)} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 2.6 $|U^0| = 0$.

When $n \leq 5$, we can redivide the Q_n with $|U^0|, |U^1| \geq 1$. We will prove this case with $n \geq 6$. Let (s_2, b_1) and (t_2, b_2) be two fault-free edges in Q_n^1 with $b_1', b_2' \notin F_a$. Applying Lemma 1, we can construct a Hamiltonian path $P(b_1', b_2')$ of $Q_n^0 - F_a - F_v^0$. We can also construct a Hamiltonian path $P(s_1, t_1)$ of $Q_n^1 - F_v^1 - \{s_2, t_2, b_1, b_2\}$ for $n \geq 6$. Thus, $P(s_1, t_1)$ and $P\langle s_2, b_1, b_1' \xrightarrow{P(b_1',b_2')} b_2', b_2, t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 3 $F_a^i = \emptyset$ and $|F_v^i| = 2$ for some $i = 0, 1$.

Without loss of generality, we can assume that $i = 1$.

Case 3.1 $|U^0| = 4$.

Applying Lemma 1, we can construct a Hamiltonian cycle ,without loss of generality, we can assume that cycle $P\langle s_2, b_3 \xrightarrow{P(b_3,t_1)} t_1, b_2 \xrightarrow{P(b_2,t_2)} t_2, b_4 \xrightarrow{P(b_4,s_1)} s_1, b_1 \xrightarrow{P(b_1,s_2)} s_2\rangle$ of $Q_n^0 - F_a$. With the induction hypothesis, there exist two spanning disjoint paths $P(b_1', b_2')$ and $P(b_3', b_4')$ of $Q_n^1 - F_v$. Thus, $\langle s_1 \xrightarrow{P(s_1,b_4)} b_4, b_4' \xrightarrow{P(b_4',b_3')} b_3', b_3 \xrightarrow{P(b_3,t_1)} t_1\rangle$ and $P\langle s_2 \xrightarrow{P(s_2,b_1)} b_1, b_1' \xrightarrow{P(b_1',b_2')} b_2', b_2 \xrightarrow{P(b_2,t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$, as illustrated in Figure 5.

Case 3.2 $|U^0| = 3$.

Without loss of generality, we can assume that $t_2 \in Q_n^1$. Let b_1 be a fault-free vertex in Q_n^0 with $b_1' \neq F_v$ and $P(s_1, b_1) > 1$. Applying Lemma 1, we can construct a Hamiltonian path $\langle b_1 \xrightarrow{P(b_1,b_4)} b_4, s_1, b_2 \xrightarrow{P(b_2,t_1)} t_1, b_3 \xrightarrow{P(b_3,s_2)} s_2\rangle$ of $Q_n^0 - F_a$ with $b_3' \neq t_2$.

Suppose that $b_2' \neq t_2$. With the

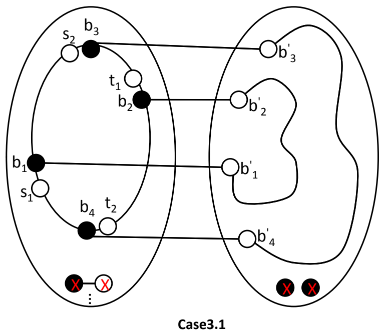

Figure 5: Illustration of case 3.1 of Theorem 2.

induction hypothesis, there exist two spanning disjoint paths $P(b_1', b_2')$ and $P(b_3', t_2)$ of $Q_n^1 - F_v$. Thus, $\langle s_1, b_4 \xrightarrow{P(b_4,b_1)} b_1, b_1' \xrightarrow{P(b_1',b_2')} b_2', b_2 \xrightarrow{P(b_2,t_1)} t_1\rangle$ and $\langle s_2 \xrightarrow{P(s_2,b_3)} b_3, b_3' \xrightarrow{P(b_3',t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Suppose that $b_2' = t_2$. With the induction hypothesis, there exist two spanning disjoint paths $P(b_3', b_1')$ and $P(b_4', t_2)$ of $Q_n^1 - F_v$. Thus, $\langle s_1, b_2 \xrightarrow{P(b_2,t_1)} t_1\rangle$ and $\langle s_2 \xrightarrow{P(s_2,b_3)} b_3, b_3' \xrightarrow{P(b_3',b_1')} b_1', b_1 \xrightarrow{P(b_1,b_4)} b_4, b_4' \xrightarrow{P(b_4',t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 3.3 $|U^0| = 2$.

Let b_1, b_2 be a fault-free vertices in Q_n^0 with $b_1', b_2' \notin F_v, t_1, t_2$.

Suppose that the two vertices of U^0 are not the same pair. Without loss of generality, we can assume that $t_1, t_2 \in Q_n^1$. Applying Lemma 2, we can construct two spanning disjoint paths $P(s_1, b_1)$ and $P(s_2, b_2)$ of $Q_n^0 - F_a$. With the induction hypothesis, there exist two spanning disjoint paths $P(b_1', t_1)$ and $P(b_2', t_2)$ of $Q_n^1 - F_v$. Thus, $\langle s_1 \xrightarrow{P(s_1,b_1)} b_1, b_1' \xrightarrow{P(b_1',t_1)} t_1\rangle$ and $\langle s_2 \xrightarrow{P(s_2,b_2)} b_2, b_2' \xrightarrow{P(b_2',t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Suppose that the two vertices of U^0 are the same pair. Without loss of generality, we can assume that $s_2, t_2 \in Q_n^1$. Applying Lemma 2, we can construct two spanning disjoint paths $P(s_1, b_1)$ and $P(b_2, t_1)$ of $Q_n^0 - F_a$. By the induction hypothesis, there exist two spanning disjoint paths $P(b'_1, b'_2)$ and $P(s_2, t_2)$ of $Q^1_n - F_v$. Thus, $\langle s_1 \xrightarrow{P(s_1,b_1)} b_1, b'_1 \xrightarrow{P(b'_1,b'_2)} b'_2, b_2 \xrightarrow{P(b_2,t_1)} t_1\rangle$ and $P(s_2, t_2)$ are two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 3.4 $|U^0| = 1$.
Without loss of generality, we can assume that $t_1, s_2, t_2 \in Q^1_n$. Let b_1 be a fault-free vertex in Q^0_n with $b'_1 \notin \{t_1, s_2, t_2\}$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, b_1)$ of $Q^0_n - F_a$. With the induction hypothesis, there exist two spanning disjoint paths $P(b'_1, t_1)$ and $P(s_2, t_2)$ of $Q^1_n - F_v$. Thus, $\langle s_1 \xrightarrow{P(s_1,b_1)} b_1, b'_1 \xrightarrow{P(b'_1,t_1)} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 3.5 $|U^0| = 0$.
With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q^1_n - F_v$ with $|P(s_1, t_1)| \geq |P(s_2, t_2)|$. Since $(2^{n-1} - 4)/2 > 2 \cdot 2 \cdot |F_a|$, there exists an edge (b_1, w_1) of $P(s_1, t_1)$ with $b'_1, w'_1 \notin F_a$ and $b_1 \in V_b$. Applying Lemma 1, we can construct a Hamiltonian path $P(b'_1, w'_1)$ of $Q^0_n - F_a$. Thus, $\langle s_1 \xrightarrow{P(s_1,b_1)} b_1, b'_1 \xrightarrow{P(b'_1,w'_1)} w'_1, w_1 \xrightarrow{P(w_1,t_1)} t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 4 $F^0_a \neq \emptyset, F^1_a \neq \emptyset$ and $|F^i_v| = 2$ for some $i = 0, 1$.
Without loss of generality, we can assume that $i = 0$.

Case 4.1 $|U^0| = 4$.
The proof of this case is similar to Case 3.5.

Case 4.2 $|U^0| = 3$.
The proof of this case is similar to Case 3.4.

Case 4.3 $|U^0| = 2$.
The proof of this case is similar to Case 3.3.

Case 4.4 $|U^0| = 1$.
Without loss of generality, we can assume that $t_1, s_2, t_2 \in Q^1_n$. Let (b_1, t_1) be a fault-free edges in Q^1_n with $b'_1 \notin F^0_a$. Let w_1, w_2 be fault-free vertices in Q^0_n with $w'_1, w'_2 \notin F^1_a$. With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, b'_1)$ and $P(w_1, w_2)$ of $Q^0_n - F^0_a - F_v$. Applying Lemma 2, we can construct two spanning disjoint paths $P(s_2, w'_1)$ and $P(w'_2, t_2)$ of $Q^1_n - F^1_a - \{t_1, b_1\}$. Thus, $\langle s_1 \xrightarrow{P(s_1,b'_1)} b'_1, b_1, t_1\rangle$ and $\langle s_2 \xrightarrow{P(s_2,w'_1)} w'_1, w_1 \xrightarrow{P(w_1,w_2)} w_2, w'_2 \xrightarrow{P(w'_2,t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 4.5 $|U^0| = 0$.
When $n \leq 6$, we can rearrange two vertices of U in different sub-cube. Thus, we only need to prove this case for $n \geq 7$. Let (s_1, b_1) and (s_2, b_2) be two fault-free edges in Q^1_n with $b'_1, b'_2 \notin F^0_a$. Let w_1, w_2 be fault-free vertices of in Q^0_n with $w'_1, w'_2 \notin F^1_a$. With the induction hypothesis, there exist two spanning disjoint paths $P(b'_1, w_1)$ and $P(b'_2, w_2)$ of $Q^0_n - F^0_a - F_v$. Applying Lemma 2, we can construct two spanning disjoint paths $P(w'_1, t_1)$ and $P(w'_2, t_2)$ of $Q^1_n - F^1_a - s_1, s_2, b_1, b_2$. Thus, $\langle s_1, b_1, b'_1 \xrightarrow{P(b'_1,w_1)} w_1, w'_1 \xrightarrow{P(w'_1,t_1)} t_1\rangle$ and $\langle s_2, b_2, b'_2 \xrightarrow{P(b'_2,w_2)} w_2, w'_2 \xrightarrow{P(w'_2,t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 5 $F^0_a \neq \emptyset, F^1_a \neq \emptyset$ and $|F^0_v| = |F^0_v| = 1$.
Without loss of generality, we can assume that $|U^0| \geq |U^1|$.

Case 5.1 $|U^0| = 4$, $|F^1_a| = 1$.
Let $F^1_a = \{f_b, f_w\}$. Suppose that at least one vertex of U is not adjacent to f'_w. Without loss of generality, we can assume that s_2 is not adjacent to f'_w. Let (s_2, b_1) be a fault-free edge for $b'_1 \notin F^1_a$. With the induction hypothesis, there exist two spanning disjoint paths $P(s_1, t_1)$ and $\langle s_2, b_2 \xrightarrow{P(b_2,t_2)} t_2\rangle$ of $Q^0_n - F^0_a - F^0_v - \{b_1\}$. Applying Lemma 1, we can construct a Hamiltonian path $P(b'_1, b'_2)$ of $Q^1_n - F^1_a - F^1_v$. Thus, $P(s_1, t_1)$ and $\langle s_2, b_1, b'_1 \xrightarrow{P(b'_1,b'_2)} b'_2, b_2 \xrightarrow{P(b_2,t_2)} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.
Suppose every vertex of U is adjacent to f'_w. Applying Lemma 1, we can construct a Hamiltonian path $\langle t_1, b_3 \xrightarrow{P(b_3,s_2)} s_2, b_4 \xrightarrow{P(b_4,t_2)} t_2\rangle$ of $Q^0_n - F^0_a - F^0_v - \{s_1, f'_w\}$. We can also construct a Hamiltonian path $P(b'_3, b'_4)$ of $Q^1_n - F^1_a -$

F_v^1. Thus, $\langle s_1, f'_w, t_1\rangle$ and $\langle s_2 \stackrel{P(s_2,b_3)}{\longrightarrow} b_3, b'_3 \stackrel{P(b'_3,b'_4)}{\longrightarrow} b'_4, b_4 \stackrel{P(b_4,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$, as illustrated in Figure 6.

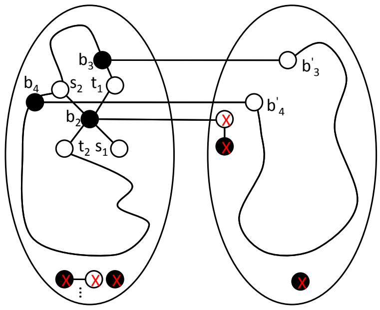

Figure 6: Illustration of case 5.1 of Theorem 2.

Case 5.2 $|U^0| = 4$, $F_a^1 \geq 2$.
Let (s_1, b_1) and (b_2, t_1) be two fault-free edges in Q_n^0 with $b'_1, b'_2 \notin F_a^1, F_v^1$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_2, t_2)$ of $Q_n^0 - F_a^0 - F_v^0 - \{s_1, t_1, b_1, b_2\}$. We can also construct a Hamiltonian path $P(b'_1, b'_2)$ of $Q_n^1 - F_a^1 - F_v^1$. Thus, $\langle s_1, b_1, b'_1 \stackrel{P(b'_1,b'_2)}{\longrightarrow} b'_2, b_2, t_1\rangle$ and $P(s_2, t_2)$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 5.3 $|U^0| = 3$.
Without loss of generality, we can assume that $t_2 \in Q_n^1$. Let (s_2, b_1) be a fault-free edges in Q_n^0 with $b'_1 \notin (F_a^1 \cup \{t_2\})$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_1, t_1)$ of $Q_n^0 - F_a^0 - F_v^0 - \{s_2, b_1\}$. We can also construct a Hamiltonian path $P(b'_1, t_2)$ of $Q_n^1 - F_a^1 - F_v^1$. Thus, $P(s_1, t_1)$ and $P\langle s_2, b_1, b'_1 \stackrel{P(b'_1,t_2)}{\longrightarrow} t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 5.4 $|U^0| = 2$ and the two vertices of U^0 are not the same pair.
Without loss of generality, we can assume that $t_1, t_2 \in Q_n^1$. Let (s_1, b_1) be a fault-free edge in Q_n^0 with $b'_1 \notin (F_a^1 \cup \{t_1, t_2\})$ and (w'_1, t_2) be a fault-free edge in Q_n^1 with $w_1 \notin (F_a^0 \cup \{s_1, s_2\})$. Applying Lemma 1, we can construct a Hamiltonian path $P(s_2, w_1)$ of $Q_n^0 - F_a^0 - F_v^0 - \{s_1, b_1\}$ and a Hamiltonian path $P(b'_1, t_1)$ of $Q_n^1 - F_a^1 - F_v^1 - \{w'_1, t_2\}$. Thus, $\langle s_1, b_1, b'_1 \stackrel{P(b'_1,t_1)}{\longrightarrow} t_1\rangle$ and $\langle s_2 \stackrel{P(s_2,w_1)}{\longrightarrow} w_1, w'_1, t_2\rangle$ are the two spanning disjoint paths of $Q_n - F_a - F_v$.

Case 5.5 $|U^0| = 2$and the two vertices of U^0 are the same pair.
The proof of this case is similar to Case 2.4.

□

Let x be a white vertex of Q_n and (x, x_i) be an edge of Q_n for $1 \leq i \leq n$. Let $x_4, \cdots, x_n \in F_a$ and $s_1, s_2, t_1, t_2 \in V_w$. There are not two spanning disjoint paths $P(s_1, t_1)$ and $P(s_2, t_2)$ of $Q_n - F_a - \{x_2, x_3\}$ with $x \notin \{s_1, t_1, s_2, t_2\}$. Thus, the result of Theorem 2 is optimal.

4 Conclusion

In this paper, we show two optimal results:

- Q_n is f-adjacency $(n-3-f)$ edges property 2H with same color for $f \leq n-4$,
- Q_n is f-adjacency property 2H adding two black faulty nodes for $f \leq n-4$.

We can investigate more Hamiltonian fault tolerance properties with these results.

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